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A weight statistic and partial order on products of m-cycles

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ABSTRACT

R. S. Deodhar and M. K. Srinivasan defined a weight statistic on the set of involutions in the symmetric group and proved several results about the properties of this weight. These results include a recursion for a weight generating function, that the weight provides a grading for the set of fixed-point free involutions under a partial order related to the Bruhat partial order, and that this graded poset is EL-shellable and its order complex triangulates a ball. We extend the definition of weight to products of disjoint m-cycles in the symmetric group, and we generalize all of the results of Deodhar and Srinivasan just mentioned to the case of any $m \geq 2$.

1. Introduction

Let m, n be integers, with $2 \leq m \leq n$, let S_n be the symmetric group on $[n] = \{1, 2, \ldots, n\}$, and let $\delta \in S_n$ be a product of disjoint m-cycles. In particular, if m is prime, then δ is just an element of order m in S_n (or the identity, if it is the product of zero m-cycles). Writing δ in cycle notation, suppose we have δ is a product of k disjoint m-cycles (so $mk \leq n$), so that

$$\delta = (a_{1,1}\ a_{1,2} \cdots\ a_{1,m})(a_{2,1}\ a_{2,2}\ \cdots\ a_{2,m}) \cdots (a_{k,1}\ a_{k,2}\ \cdots\ a_{k,m}). \tag{1.1}$$

Further, suppose that for each $i = 1, 2, \ldots, k$, $a_{i,1} < a_{i,j}$ for $j = 2, 3, \ldots, m$, and $a_{1,1} < a_{2,1} < \cdots < a_{k,1}$, and we then say that δ is in *standard form*. Let $J^{(m)}(n)$ denote the collection of all products of disjoint m-cycles in S_n, and let $J^{(m)}(n, k)$ denote the collection of all products of k disjoint m-cycles in S_n, so that $\delta \in J^{(m)}(n, k)$ in (1.1). Given $\delta \in J^{(m)}(n, k)$ in standard form as in (1.1), define span(δ) as

$$\mathrm{span}(\delta) = \sum_{i=1}^{k} \sum_{j=2}^{m} (a_{i,j} - a_{i,1} - 1).$$

For example, suppose $\delta \in J^{(3)}(9, 3)$, where $\delta = (1\ 6\ 9)(2\ 7\ 4)(3\ 5\ 8)$. Then

$$\mathrm{span}(\delta) = (5 - 1) + (8 - 1) + (5 - 1) + (2 - 1) + (2 - 1) + (5 - 1) = 21.$$

Given an m-cycle $(a_1\ a_2 \ldots a_m) \in S_n$ in standard form, draw its *arc diagram* by drawing, along a line containing points labeled from $[n]$, an arc for each pair (a_1, a_j), $j = 2, \ldots, m$, where the arc (a_1, a_l) is drawn under (a_1, a_j) when $l > j$. When $a_j > a_l$ and $j > l$, then the arcs (a_1, a_j) and (a_1, a_l) intersect in the arc diagram, which we call an *internal crossing* of the m-cycle. That is, the number of internal crossings of the m-cycle $(a_1\ a_2 \ldots a_m)$ is equal to the number of pairs (a_i, a_j) from the sequence $a_2, a_3, \ldots, a_m$, which satisfy $i < j$ and $a_i < a_j$, which we call *ascents* of this sequence (which are also known as non-inversions). In Figs. 1 and 2, for example, we show the arc diagrams for the 5-cycles (1 4 5 3 2) and (1 2 4 5 3), with the internal crossings circled.

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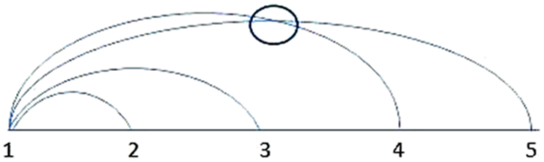

Fig. 1. Arc diagram and internal crossing of (1 4 5 3 2).

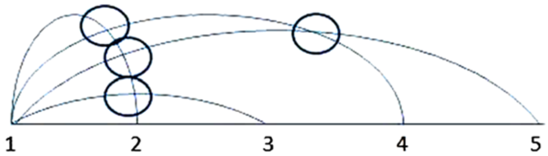

Fig. 2. Arc diagram and internal crossings of (1 2 4 5 3).

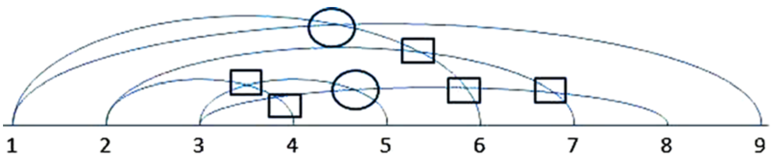

Fig. 3. Arc diagram and all crossings of (1 6 9) (2 7 4) (3 5 8).

If $\delta \in J^{(m)}(n, k)$, the arc diagram for δ is drawn by drawing the arc diagram for each of the k disjoint m-cycles of δ. There may be intersections of the arcs from different m-cycles of δ, which we call *external crossings* of δ. Let $C_{\rm in}(\delta)$ denote the total number of internal crossings in the arc diagram of δ, and $C_{\rm ex}(\delta)$ denote the total number of external crossings in the arc diagram of δ. The *crossing number* of δ, $C(\delta)$, is then defined as $C(\delta) = C_{\rm ex}(\delta) + C_{\rm in}(\delta)$. In Fig. 3, we show the arc diagram for $\delta = (1\ 6\ 9)(2\ 7\ 4)(3\ 5\ 8) \in J^{(3)}(9, 3)$, with internal crossings in circles and external crossings in boxes.

Now, given any $\delta \in J^{(m)}(n)$, we define the *weight* of δ, which we denote $\mathrm{wt}_m(\delta)$, as

$$\mathrm{wt}_m(\delta) = \mathrm{span}(\delta) - C(\delta) = \mathrm{span}(\delta) - C_{\rm ex}(\delta) - C_{\rm in}(\delta).$$

So, for $\delta = (1\ 6\ 9)(2\ 7\ 4)(3\ 5\ 8)$, since $\mathrm{span}(\delta) = 21$ and $C(\delta) = 7$ from Fig. 3, then $\mathrm{wt}_3(\delta) = 14$. We note that for $m = 2$, our definition of weight coincides precisely with that from [5], since if $\delta \in S_n$ is an involution, then $C_{\rm in}(\delta) = 0$.

Given integers $n, k \geq 0$, $m \geq 2$, with $mk \leq n$, define the *weight generating function*, denoted $j_q^{(m)}(n, k)$, to be the following polynomial in an indeterminate q:

$$j_q^{(m)}(n, k) = \sum_{\delta \in J^{(m)}(n,k)} q^{\mathrm{wt}_m(\delta)}.$$

In Section 2, we prove a recursive relation on the weight generating function in terms of n and k (Theorem 2.1), and we use it to compute an exact formula (Corollary 2.1) for $j_q^{(m)}(mn, n)$, the weight generating function for the set of products of disjoint m-cycles in S_{mn} which are fixed-point free. We use the notation $F^{(m)}(mn) = J^{(m)}(mn, n)$ for the set of fixed-point free products of disjoint m-cycles in S_{mn}.

In Section 3, we introduce the Bruhat order (sometimes called the *strong* Bruhat order) on the symmetric group S_n, which makes S_n a graded poset with grading given by the number of inversions of a permutation. We consider a specific subset $E(n_m)$ of a permutation group $S(n_m)$ (identified with S_{mn}), and the set $E(n_m)$ is in bijection with the set $F^{(m)}(mn)$ of fixed-point free products of disjoint m-cycles in S_{mn}. In Proposition 3.1, we show that an explicit bijection ϕ defined between these two sets maps the weight wt_m of a permutation in $F^{(m)}(mn)$ to the number of inversions of the permutation in $E(n_m)$.

Now let $\delta, \pi \in F^{(m)}(mn)$, and suppose that

$$\delta = (a_{1,1}\ a_{1,2} \cdots a_{1,m})(a_{2,1}\ a_{2,2}\ \cdots\ a_{2,m}) \cdots (a_{n,1}\ a_{n,2}\ \cdots\ a_{n,m})$$

is in standard form. Then π is obtained from δ by an *interchange* if one of the following holds:

(i) There is some i, $1 \leq i \leq n$, and some j, l, $2 \leq j, l \leq m$, such that the standard form of π is obtained by interchanging $a_{i,j}$ and $a_{i,l}$ in δ.
(ii) There are some i, j, $1 \leq i < j \leq n$, and some l, $2 \leq l \leq m$, such that the standard form of π is obtained by interchanging $a_{i,l}$ and $a_{j,1}$ in δ.
(iii) There are some i, j, l, h, $2 \leq l, h \leq m$, $l \neq h$, $1 \leq i < j \leq n$, such that the standard form of π is obtained by interchanging $a_{i,l}$ and $a_{j,h}$ in δ.

If π is obtained from δ by an interchange, then such an interchange is *weight increasing* if $\mathrm{wt}_m(\delta) < \mathrm{wt}_m(\pi)$. We define a relation on the set $F^{(m)}(mn)$ as follows. If $\delta, \sigma \in F^{(m)}(mn)$, then we write $\delta \preceq \sigma$ if σ is obtained by δ by a sequence of zero or more weight increasing interchanges. In Proposition 3.2, we show that the bijection $\phi : F^{(m)}(mn) \to E(n_m)$, which we show in Proposition 3.1 carries wt_m to the number of inversions, is an isomorphism of posets, where $F^{(m)}(mn)$ is given the partial order $\preceq$ just defined, and $E(n_m)$ is given the Bruhat order. This is not quite enough to conclude that $F^{(m)}(mn)$ is a graded poset with rank given by wt_m, since it is not apparent that the sub-poset $E(n_m)$ of $S(n_m)$ is graded by the number of inversions. This requires the notion of EL-labelings, which we introduce in Section 4.

After defining EL-labelings and EL-shellable graded posets in Section 4, and the EL-labeling defined on S_n with respect to the Bruhat order, we show that $E(n_m)$ inherits this EL-labeling from $S(n_m)$ in Proposition 4.1. It follows that $E(n_m)$ is an EL-shellable graded poset. We immediately obtain our next main result, Theorem 4.1, which states that $(F^{(m)}(mn), \preceq)$ is an EL-shellable graded poset, of rank $\frac{(m-1)n(mn-2)}{2}$, with grading given by wt_m, and with explicit rank generating function found in Corollary 2.1. Finally, we introduce the notion of the order complex of a poset, and we show in Theorem 4.2 that the order complex of $F^{(m)}(mn)$, less its maximal and minimal elements, triangulates a ball of dimension $\frac{(m-1)n(mn-2)}{2} - 2$.

All of the results mentioned above are generalizations of results obtained by Deodhar and Srinivasan [5]. We adapt our arguments from those given by Deodhar and Srinivasan, and throughout we specify precisely which results and arguments from [5] are being generalized. We order material somewhat differently than in [5], partially to stress which parts of this paper require more work than the case $m = 2$. That is, the paper is ordered roughly in order from results requiring more argument than in [5] to results requiring essentially the same. The main idea in this paper is finding the "right" generalization of weight for products of disjoint m-cycles, for any $m \geq 2$. The fact that our definition fits the bill reveals itself in the proofs of Theorem 2.1 and Proposition 3.1, which require a more intricate proof than their $m = 2$ counterparts. Proposition 3.2 then requires more cases to check than the $m = 2$ case. The arguments given in Section 4 then go through nearly the same as in the $m = 2$ case, with only cosmetic changes.

Other than their results which inspired this paper, Deodhar and Srinivasan also showed [6] that their weight defined on involutions in S_n is a specialization of the weight defined by W. P. Johnson [9,8] on set partitions. So it is reasonable to expect that our weight function on products of disjoint m-cycles should be a specialization of a more general weight than Johnson's on set partitions. It would be interesting to have such a generalization and to use it to find combinatorial applications which generalize those found by Johnson.

Can, Cherniavsky, and Twelbeck [3] have studied the Bruhat order on fixed-point free involutions, and in particular have shown that the poset of fixed-point free involutions studied by Deodhar and Srinivasan is a sub-poset of the fixed-point free involutions under the Bruhat order [3, Theorem 10]. It seems to be a worthwhile question to understand the (very likely more complicated) relationship between the poset $F^{(m)}(mn)$ we study here and the Bruhat poset of fixed-point free products of disjoint m-cycles in S_{mn} when $m > 2$.

Finally, we point out that after the original version of this paper was written, many of its results were further generalized by Can and Cherniavsky [2]. We make remarks on these results at the end of Section 3.

2. Recursion for the generating function

As in the introduction, let $J^{(m)}(n, k)$ denote the set of permutations in S_n which are the product of k disjoint m-cycles. A counting argument gives the recursive relation

$$|J^{(m)}(n+1, k)| = |J^{(m)}(n, k)| + n(n-1)\cdots(n-m+2)|J^{(m)}(n-m+1, k-1)|.$$

The main result of this section is a refinement of this recursion in terms of the weight function wt_m defined in the introduction. In particular, Theorem 2.1 gives a recursive relation for the generating function $j_q^{(m)}(n, k) = \sum_{\delta \in J^{(m)}(n,k)} q^{\mathrm{wt}_m(\delta)}$, giving a q-analog of the recursion for $|J^{(m)}(n, k)|$ above. This generalizes [5, Proposition 2.1].

Before giving the result, we clarify the following notation. For the indeterminate q, define $[n]_q = q^{n-1} + q^{n-2} + \cdots + q + 1$ for any $n \geq 1$. If $\delta \in J^{(m)}(n, k)$ and the m-cycle $(a_{i,1}\ a_{i,2} \ldots a_{i,m})$ is one of the cycles in δ in standard form, then we write $(a_{i,1}\ a_{i,2} \ldots a_{i,m})|\delta$.

Theorem 2.1. *The following recursion holds for the generating function $j_q^{(m)}(n, k)$:*

$$j_q^{(m)}(n+1, k) = j_q^{(m)}(n, k) + [n]_q[n-1]_q \cdots [n-m+2]_q j_q^{(m)}(n-m+1, k-1).$$

Proof. We begin by obtaining a bijection

$$\Theta : J^{(m)}(n+1, k) \to J^{(m)}(n, k) \cup \left([n] \times [n-1] \times \cdots \times [n-m+2] \times J^{(m)}(n-m+1, k-1)\right).$$

For any $\delta \in J^{(m)}(n+1, k)$, if $(1\ a_{1,2}\ a_{1,3} \cdots a_{1,m}) \nmid \delta$, that is, if $a_{1,1} \neq 1$ when δ is in standard form, then label each $a_{i,j}$ of δ by $a_{i,j} - 1$ for all $1 \leq i \leq k$, $1 \leq j \leq m$ and define the resulting element to be $\Theta(\delta) \in J^{(m)}(n, k)$.

Now consider the case that $(1\ \ a_{1,2}\ \ a_{1,3} \ldots a_{1,m})|\delta$, so $a_{1,1} = 1$ in δ in standard form. Delete the m-cycle $(1\ \ a_{1,2}\ a_{1,3} \cdots a_{1,m})$ from δ and name the resulting element $\bar{\delta}$. Now, in the set $[n+1]/\{1, a_{1,2}, a_{1,3}, \ldots, a_{1,m}\}$, relabel the elements of this set, in increasing order, by $1, 2, \ldots, n-m+1$. Apply this relabeling to the entries of $\bar{\delta}$ and name the resulting element $\delta' \in J^{(m)}(n-m+1, k-1)$. For each $a_{1,l}$, $2 \le l \le m$, define

$$f(a_{1,l}) = \text{the number of } a_{1,j} \text{ such that } 1 < j < l \text{ and } a_{1,j} < a_{1,l}.$$

That is, $f(a_{1,l})$ is equal to the number of arcs in the arc diagram of the cycle $(a_{1,1} \ldots a_{1,m})$ which are above the arc from 1 to $a_{1,l}$ and which intersect with the arc between $a_{1,1} = 1$ and $a_{1,l}$. So, $\sum_{l=2}^{m} f(a_{1,l})$ is the total number of internal crossings of this cycle, which is also the number of ascents in the sequence $a_{1,2}, \ldots, a_{1,m}$. Note that we have $a_{1,l} > f(a_{1,l}) + 1$, and if $a_{1,l} > n-l+2$, say $a_{1,l} = n-l+2+j$ for some $j \ge 1$, then it follows that $f(a_{1,l}) \ge j-1$. That is, $a_{1,l} - 1 - f(a_{1,l}) \in [n-l+2]$. With these definitions, define $\Theta(\delta)$ in the case that $(1\ a_{1,2}\ a_{1,3} \ldots a_{1,m})|\delta$ by

$$\Theta(\delta) = (a_{1,2}-1, a_{1,3}-1-f(a_{1,3}), \ldots, a_{1,m}-1-f(a_{1,m}), \delta').$$

To see that Θ is indeed a bijection, it suffices to show that it is injective. If $\Theta(\delta_1) = \Theta(\delta_2) \in J^{(m)}(n, k)$, it follows immediately that $\delta_1 = \delta_2$, so suppose $\Theta(\delta_1) = \Theta(\delta_2) \notin J^{(m)}(n, k)$. So,

$$\begin{aligned}\Theta(\delta_1) &= (a_{1,2}-1, a_{1,3}-1-f(a_{1,3}), \ldots, a_{1,m}-1-f(a_{1,m}), \delta_1') \\ &= (b_{1,2}-1, b_{1,3}-1-f(b_{1,3}), \ldots, b_{1,m}-1-f(b_{1,m}), \delta_2') = \Theta(\delta_2).\end{aligned}$$

Then $a_{1,2} = b_{1,2}$ and $a_{1,3} - f(a_{1,3}) = b_{1,3} - f(b_{1,3})$. If $f(a_{1,3}) \neq f(b_{1,3})$, say $f(a_{1,3}) > f(b_{1,3})$, then we must have $f(a_{1,3}) = 1$ and $f(b_{1,3}) = 0$, so $a_{1,3} - b_{1,3} = 1$. This is impossible, since we must also have $b_{1,3} < b_{1,2} = a_{1,2} < a_{1,3}$. Thus $f(a_{1,3}) = f(b_{1,3})$ and $a_{1,3} = b_{1,3}$. By induction, suppose $j < m$ and for each $l \le j$ we have $a_{1,l} = b_{1,l}$ and so $f(a_{1,l}) = f(b_{1,l})$. If $f(a_{1,j+1}) \neq f(b_{1,j+1})$, say $f(a_{1,j+1}) = f(b_{1,j+1}) + k$, $k > 0$, then $a_{1,j+1} = b_{1,j+1} + k$ since $a_{1,j+1} - f(a_{1,j+1}) = b_{1,j+1} - f(b_{1,j+1})$. However, we then have

$$k = \text{the number of } a_{1,l} = b_{1,l} \text{ such that } 2 \le l \le j \text{ and } b_{1,j+1} < a_{1,l} < a_{1,j+1},$$

which is impossible. Thus $a_{1,l} = b_{1,l}$ for each l, and since $\delta_1' = \delta_2'$, we have $\delta_1 = \delta_2$.

Let $\delta \in J^{(m)}(n+1, k)$. If $(1\ a_{1,2}\ a_{1,3} \ldots a_{1,m}) \nmid \delta$, it follows that $\mathrm{wt}_m(\delta) = \mathrm{wt}_m(\Theta(\delta))$. If $(1\ a_{1,2}\ a_{1,3} \ldots a_{1,m})|\delta$, we claim that

$$\mathrm{wt}_m(\delta) = \mathrm{wt}_m(\delta') + a_{1,2} - 2 + a_{1,3} - 2 - f(a_{1,3}) + \cdots + a_{1,m} - 2 - f(a_{1,m}).$$

By considering the bijection Θ just constructed, along with the powers of q which occur on both sides of the desired recursion, one sees that proving this claim finishes the proof.

Consider the arc diagram for δ, and define A_k to be the number of arcs of δ without 1 as an endpoint which cross exactly k arcs of δ which do have endpoint 1. It follows that we have

$$C_{\mathrm{ex}}(\delta) = C_{\mathrm{ex}}(\bar{\delta}) + \sum_{j=1}^{m-1} jA_j = C_{\mathrm{ex}}(\delta') + \sum_{j=1}^{m-1} jA_j.$$

We also have

$$\begin{aligned}\mathrm{span}(\delta) &= \mathrm{span}(\bar{\delta}) + a_{1,2} - 2 + a_{1,3} - 2 + \cdots + a_{1,m} - 2 \\ &= \mathrm{span}(\delta') + a_{1,2} - 2 + a_{1,3} - 2 + \cdots + a_{1,m} - 2 + \sum_{j=1}^{m-1} jA_j,\end{aligned}$$

since each arc counted by an A_j corresponds to an element of $[n+1]$ underneath an arc of $\bar{\delta}$, which is removed in the relabeling process when constructing δ'. From the definition of the function f, we also have

$$C_{\mathrm{in}}(\delta) = C_{\mathrm{in}}(\bar{\delta}) + f(a_{1,3}) + \cdots + f(a_{1,m}) = C_{\mathrm{in}}(\delta') + f(a_{1,3}) + \cdots + f(a_{1,m}).$$

Therefore, we have

$$\begin{aligned}\mathrm{wt}_m(\delta) &= \mathrm{span}(\delta) - C_{\mathrm{ex}}(\delta) - C_{\mathrm{in}}(\delta) \\ &= \mathrm{span}(\delta') + a_{1,2} - 2 + a_{1,3} - 2 + \cdots + a_{1,m} - 2 + \sum_{j=1}^{m-1} jA_j \\ &\quad - \left(C_{\mathrm{ex}}(\delta') + \sum_{j=1}^{m-1} jA_j \right) - \left(C_{\mathrm{in}}(\delta') + f(a_{1,3}) + \cdots + f(a_{1,m}) \right) \\ &= \mathrm{wt}_m(\delta') + a_{1,2} - 2 + a_{1,3} - 2 - f(a_{1,3}) + \cdots + a_{1,m} - 2 - f(a_{1,m}),\end{aligned}$$

giving the claim. □

Using Theorem 2.1, we may calculate a precise formula for the weight generating function in the fixed-point free case. For $n > 0$, define $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$, and define $[0]_q! = 1$. The following generalizes [5, Proposition 2.3].

Corollary 2.1. *For any $n \geq 0$, we have*

$$j_q^{(m)}(mn, n) = \sum_{\delta \in F^{(m)}(mn)} q^{\mathrm{wt}_m(\delta)} = \frac{[mn]_q!}{[mn]_q[m(n-1)]_q \cdots [m]_q}.$$

Proof. Since $j_q^{(m)}(0, 0) = 1$, and $j_q^{(m)}(mn-1, n) = 0$ for any $n \geq 1$, the result follows from Theorem 2.1 and induction. □

Remark. It is at this point in [5] when Deodhar and Srinivasan obtain an expansion of the q-binomial coefficient as a sum over involutions, in terms of the weight function and the regular binomial coefficients, with implications about the poset of subspaces of a finite vector space. This was the only result from [5] for which we were unable to obtain a meaningful generalization. It would be nice to have such a generalization and to understand the meaning of Theorem 2.1 in the context of finite vector spaces.

3. Bruhat order and the poset $E(n_m)$

Given any element π in the symmetric group S_n, we may write π in permutation notation, as in $\pi = \pi_1\pi_2 \cdots \pi_n$. An *inversion* of π is a pair $(i, j) \in [n] \times [n]$ such that $i < j$ and $\pi_i > \pi_j$. Let $\iota(\pi)$ denote the number of inversions of π. If $\pi' \in S_n$ such that π' is obtained from π by interchanging two π_i's in permutation notation, and $\iota(\pi) < \iota(\pi')$, then we say that π' is obtained by π by an *inversion increasing interchange*. For $\pi, \sigma \in S_n$, define $\pi \leq \sigma$ if σ can be obtained from π by a sequence of zero or more inversion increasing interchanges. This partial order is the *(strong) Bruhat order*, and it makes S_n a graded poset with grading given by ι and rank generating function $[n]_q!$ [10, Chapter 3, Exercise 183(a)].

Now, given $m \geq 2$, extend the set $[n]$ by defining, for each $l \in [n]$, elements $l_1, l_2, \ldots, l_m$, which are ordered so that $l_i < l_j$ when $i < j$, and $k_i < l_j$ when $k < l$ for any i and j. Let $[n_m]$ be the resulting linearly ordered set, that is,

$$[n_m] = \{1_1 < 1_2 < \cdots < 1_m < 2_1 < \cdots < 2_m < \cdots < n_1 < \cdots < n_m\}.$$

Then the symmetric group $S(n_m)$ on $[n_m]$ may be identified with S_{mn}, and $S(n_m)$ is a graded poset under the Bruhat order.

Define a subset $E(n_m) \subset S(n_m)$ as follows. Let $\pi \in S(n_m)$ be written in permutation form, $\pi = \pi_{1_1}\pi_{1_2} \cdots \pi_{n_m}$. Then $\pi \in E(n_m)$ if and only if, in the permutation form of π, k_1 is to the left of l_1 whenever $k < l$, and k_1 is to the left of k_j for any k and any $j \geq 2$. For example, $1_1 1_2 2_1 2_2 2_3 1_3 \in E(2_3)$, while $1_1 1_2 1_3 2_3 2_1 2_2 \notin E(2_3)$.

We now define a map $\phi : F^{(m)}(mn) \to E(n_m)$ in the following way. Let $\delta \in F^{(m)}(mn)$ be written in standard form,

$$\delta = (a_{1,1}\ a_{1,2} \cdots\ a_{1,m})(a_{2,1}\ a_{2,2}\ \cdots\ a_{2,m}) \cdots (a_{n,1}\ a_{n,2}\ \cdots\ a_{n,m}),$$

where $a_{1,1} = 1$. Starting with the identity in S_{mn} in permutation form, $123 \cdots (mn)$, replace $a_{l,j}$ with l_j and define the resulting permutation in $S(n_m)$ to be $\phi(\delta)$. For example, if $\delta = (1\ 6\ 9)(2\ 7\ 3)(4\ 5\ 8)$, then $\phi(\delta) = 1_1 2_1 2_3 3_1 3_2 1_2 2_2 3_3 1_3$. The fact that $\phi(\delta) \in E(n_m)$ for any $\delta \in F^{(m)}(mn)$ follows from the definitions of standard form and the set $E(n_m)$. A direct counting argument gives $|F^{(m)}(mn)| = |E(n_m)|$, and since ϕ is injective by construction, then ϕ is a bijection. Moreover, the map ϕ carries the weight of δ to the number of inversions of $\phi(\delta)$, as we see next. We note that $E(n_2)$ is exactly the set $E(\bar{n})$ defined by Deodhar and Srinivasan if we change each i_1 into i and i_2 into $\bar{i}$, our map ϕ generalizes their bijection between fixed-point free involutions in S_{2n} and $E(\bar{n})$, and the following is a generalization of [5, Proposition 3.3].

Proposition 3.1. *For any $\delta \in F^{(m)}(mn)$, we have $\mathrm{wt}_m(\delta) = \iota(\phi(\delta))$.*

Proof. The proof is by induction on n. For the case $n = 1$, let $\delta = (a_{1,1}\ a_{1,2} \ldots a_{1,m}) \in F^{(m)}(m)$ (where $a_{1,1} = 1$). Then $\mathrm{span}(\delta) = \sum_{j=1}^{m-2} j = (m-1)(m-2)/2$, $C_{\mathrm{ex}}(\delta) = 0$, and $C_{\mathrm{in}}(\delta)$ is the number of ascents in the sequence $a_{1,2}, \ldots, a_{1,m}$, and $\mathrm{wt}_m(\delta) = ((m-1)(m-2)/2) - C_{\mathrm{in}}(\delta)$. Then $\mathrm{wt}_m(\delta) = ((m-1)(m-2)/2) - C_{\mathrm{in}}(\delta)$ is the number of pairs $(a_{1,i}, a_{1,j})$ with $i < j$ and $a_{1,i} > a_{1,j}$, or non-ascents, in the sequence $a_{1,2}, \ldots, a_{1,m}$. Now let $\phi(\delta) = \pi = \pi_{1_1}\pi_{1_2} \cdots \pi_{1_m}$ (where $\pi_{1_1} = 1_1$), and consider $\iota(\pi)$. Note that $\pi_{1_i} = 1_j$ if and only if $a_{1,j} = i$ by the definition of ϕ. That is, $\pi^{-1}(1_j) = 1_i$ if and only if $a_{1,j} = i$, so that the number of inversions of π^{-1} is exactly the number of non-ascents in the sequence $a_{1,2}, \ldots, a_{1,m}$. That is, $\iota(\pi^{-1}) = \mathrm{wt}_m(\delta)$. But $\iota(\pi) = \iota(\pi^{-1})$, so $\iota(\pi) = \mathrm{wt}_m(\delta)$.

Now consider some $n > 1$ under the assumption that the statement holds true for $n-1$. Let $\delta \in F^{(m)}(mn)$, where δ in standard form is

$$\delta = (a_{1,1}\ a_{1,2} \ldots a_{1,m}) \cdots (a_{n,1}\ a_{n,2} \ldots a_{n,m}),$$

and let $\pi = \phi(\delta) \in E(n_m)$. As in the proof of Theorem 2.1, form $\delta' \in F^{(m)}(m(n-1))$ by deleting $(a_{1,1} \cdots a_{1,m})$ and relabeling (note that $a_{1,1} = 1$ necessarily here). Then, as we showed, we have

$$\mathrm{wt}_m(\delta) = \mathrm{wt}_m(\delta') + a_{1,2} - 2 + a_{1,3} - 2 - f(a_{1,3}) + \cdots + a_{1,m} - 2 - f(a_{1,m}),$$

where $f(a_{1,l})$ is the number of $a_{1,j}$ such that $1 < j < l$ and $a_{1,j} < a_{1,l}$.

Now let $\pi' = \phi(\delta') \in E((n-1)_m)$, so we have $\iota(\pi') = \mathrm{wt}_m(\delta')$ by the induction hypothesis. Using the definitions of δ' and ϕ, we obtain π' from π as follows. If $\pi = \pi_{1_1}\pi_{1_2}\cdots\pi_{m_n}$, then delete $1_1, 1_2, \ldots, 1_m$, and then replace each remaining i_j with $(i-1)_j$. For example, if $\pi = 1_1 2_1 2_3 3_1 3_2 1_2 2_2 3_3 1_3$, then $\pi' = 1_1 1_3 2_1 2_2 1_2 2_3$. Then, every inversion of π' corresponds to an inversion of π, and all other inversions of π are the result of the positioning of $1_2, \ldots, 1_m$, in π. In particular, 1_l is in the $a_{1,l}$-th position of the string $\pi_{1_1}\pi_{1_2}\cdots\pi_{m_n}$, and 1_l forms an inversion with any element of this string to its left, except for any 1_j such that $j < l$. That is, if we define, for each $l \geq 2$,

$$g(1_l) = \text{the number of } 1_j \text{ such that } 1 < j < l \text{ and } \pi^{-1}(1_j) < \pi^{-1}(1_l),$$

then the number of inversions of π which include 1_l is exactly $a_{1,l} - 2 - g(1_l)$. It follows from the definition of ϕ that we then have $g(1_l) = f(a_{1,l})$, so that we finally have

$$\begin{aligned}\iota(\pi) &= \iota(\pi') + a_{1,2} - 2 + a_{1,3} - 2 - g(1_3) + \cdots + a_{1,m} - 2 - g(1_m)\\ &= \mathrm{wt}_m(\delta') + a_{1,2} - 2 + a_{1,3} - 2 - f(a_{1,3}) + \cdots + a_{1,m} - 2 - f(a_{1,m}) = \mathrm{wt}_m(\delta),\end{aligned}$$

yielding the result. □

Now consider the partial order $\preceq$ on $F^{(m)}(mn)$ defined in Section 1. The following result is analogous to [5, Proposition 3.4].

Proposition 3.2. *The map $\phi : F^{(m)}(mn) \to E(n_m)$ is an order isomorphism, mapping the partial order $\preceq$ to the Bruhat order.*

Proof. We first show that ϕ preserves order. Let $\delta \in F^{(m)}(mn)$, and let δ be in standard form as $\delta = (a_{1,1}\ldots a_{1,m})\cdots(a_{n,1}\ldots a_{n,m})$. Suppose that $\tau \in F^{(m)}(mn)$ and τ is obtained by δ by an interchange. If τ in standard form is obtained from δ by exchanging $a_{i,j}$ and $a_{i,l}$, where $1 \leq i \leq n$, and $2 \leq j, l \leq m$, then $\phi(\tau)$ is obtained from $\phi(\delta)$ by exchanging i_j with i_l. If τ is obtained from δ by exchanging $a_{i,l}$ and $a_{j,1}$ for some $1 \leq i < j \leq n$ and $2 \leq l \leq m$, then $\phi(\tau)$ is obtained from $\phi(\delta)$ by exchanging i_l and j_1. If τ is obtained from δ by exchanging $a_{i,l}$ and $a_{j,h}$, where $l \neq h$, $2 \leq l, h \leq m$, and $1 \leq i < j \leq n$, then $\phi(\tau)$ is obtained from $\phi(\delta)$ by exchanging i_l and j_h. Now, if $\delta \preceq \sigma$ for some $\sigma \in F^{(m)}(mn)$, then σ is obtained from δ by some number of such interchanges which are weight increasing. Since ϕ maps the weight to the number of inversions by Proposition 3.1, then $\phi(\sigma)$ is obtained from $\phi(\delta)$ by some sequence of inversion increasing interchanges, that is, $\phi(\delta) \leq \phi(\sigma)$.

Next we show that ϕ^{-1} is order preserving. To make notation a bit more flexible, we will identify the linearly ordered set $[n_m]$ with $\{1, 2, \ldots, mn\}$, when they appear as indices in $\pi \in S(n_m)$. That is, if $\pi \in S(n_m)$ with $\pi = \pi_{1_1}\pi_{1_2}\cdots\pi_{n_m}$, then we will also write $\pi = \pi_1\pi_2\cdots\pi_{mn}$. Let $\pi, \sigma \in E(n_m)$, and suppose $\pi < \sigma$, with $\iota(\sigma) = \iota(\pi) + 1$. Let $\pi = \pi_1\pi_2\cdots\pi_{mn}$, and suppose σ is obtained from π by exchanging π_i and π_j, where $i < j$ and $\pi_i < \pi_j$. Let $\pi_i = k_l$ and $\pi_j = h_t$ for some $k_l, h_t \in [n_m]$. Write $\phi^{-1}(\pi) = (a_{1,1}\ldots a_{1,m})\cdots(a_{n,1}\ldots a_{n,m})$ in standard form. To show that ϕ^{-1} is order preserving, it is enough to show that when exchanging $a_{k,l}$ and $a_{h,t}$ in $\phi^{-1}(\pi)$, the result, which is $\phi^{-1}(\sigma)$, is again in standard form. By Proposition 3.1, then, $\mathrm{wt}_m(\phi^{-1}(\sigma)) = \mathrm{wt}_m(\phi^{-1}(\pi)) + 1$, and it will follow that $\phi^{-1}(\pi) \prec \phi^{-1}(\sigma)$.

We first claim that we must have $\pi_i = k_l \notin \{1_1, 2_1, \ldots, n_1\}$. If not, so $k_l = k_1$, then we cannot have $\pi_j = h_t \in \{1_1, 2_1, \ldots, n_1\}$, since $\pi_i < \pi_j$, and we must remain in $E(n_m)$ when exchanging π_i and π_j. On the other hand, if $\pi_i = k_1$ and $\pi_j = h_t \notin \{1_1, 2_1, \ldots, n_1\}$, then since $k_1 < h_t$, we have $k_1 < h_1 < h_t$. Since $\pi \in E(n_m)$, then k_1 is to the left of h_1, which is to the left of h_t in π. Then we cannot exchange $\pi_i = k_1$ and $\pi_j = h_t$ and remain in $E(n_m)$. Thus $k_l \notin \{1_1, 2_1, \ldots, n_1\}$.

Now assume $\pi_j = h_t \notin \{1_1, 2_1, \ldots, n_1\}$. If $\pi_i = k_l$ is such that $k = h$, then $h_l < h_t$, so $t > l \geq 2$. Then exchanging $a_{k,l}$ and $a_{h,t}$ in $\phi^{-1}(\pi)$ gives $\phi^{-1}(\sigma)$ in standard form. If $k \neq h$, then $k < h$ since $\pi_i = k_l < h_t = \pi_j$. In order to show that $\phi^{-1}(\sigma)$ is in standard form when exchanging $a_{k,l}$ and $a_{h,t}$ in $\phi^{-1}(\pi)$, we need to show that $a_{h,1} < a_{k,l} = i$, since we already know that $a_{k,1} < a_{k,l} < a_{h,t}$. If $i = a_{k,l} < a_{h,1} = y$, say, then since $a_{h,1} < a_{h,t} = j$, we have in π that $\pi_i = k_l$ is to the left of $\pi_y = h_1$, which is to the left of $\pi_j = h_t$. Then we cannot exchange π_i and π_j and remain in $E(n_m)$. So the statement follows whenever $\pi_j = h_t \notin \{1_1, 2_1, \ldots, n_1\}$.

Finally, suppose that $\pi_j \in \{1_1, 2_1, \ldots, n_1\}$, so $\pi_j = h_1$. In order to show that exchanging $a_{k,l}$ and $a_{h,1}$ in $\phi^{-1}(\pi)$ yields $\phi^{-1}(\sigma)$ in standard form, we only need to show that $a_{r,1} < a_{k,l}$ whenever $k < r < h$, since we already know that $a_{k,1} < a_{k,l} = i < j = a_{h,1}$, $a_{k,l} < a_{h,1} < a_{t,1}$ whenever $t > h$, and $a_{t,1} \leq a_{k,1} < a_{k,l}$ whenever $t \leq k$. Supposing there is an r such that $k < r < h$ and $i = a_{k,l} < a_{r,1} = x < a_{h,1} = j$, we have k_l to the left of r_1, to the left of h_1, in π. Then we cannot exchange $\pi_i = k_l$ and $\pi_j = h_1$ and remain in $E(n_m)$. We now have that $\phi^{-1}(\sigma)$ is obtained in standard form by exchanging $a_{k,l}$ and $a_{h,t}$ in $\phi^{-1}(\pi)$ in all cases. □

Now, if we knew that the grading ι on $S(n_m)$ restricted to $E(n_m)$ makes $E(n_m)$ a graded poset, we could conclude that $F^{(m)}(mn)$ was a graded poset by the previous two results. We show that $E(n_m)$ is a graded poset in the next section by considering EL-labelings.

Remark. Consider now an arbitrary permutation $\omega \in S_n$, written in cycle form, including cycles of length 1,

$$\omega = (a_{1,1}\ \cdots\ a_{k_1,1})(a_{1,2}\ \cdots\ a_{k_2,2})\cdots(a_{1,h}\ \cdots a_{k_h,h}),$$

such that $a_{1,1} < a_{1,2} < \cdots < a_{1,h}$ and $a_{1,j} < a_{i,j}$ for every $1 \leq j \leq h$ and $2 \leq i \leq k_j$. Then $\sum_{i=1}^{h} k_i = n$, so $(k_1, \ldots, k_h)$ is a composition of n, and call $(k_1, \ldots, k_h)$ the *composition type* of ω. One may consider the map $\Omega : S_n \to S_n$ defined by

$\Omega(\omega) = \pi$, where π is written in permutation (or one-line) form as $\pi = a_{1,1} \cdots a_{k_1,1} a_{1,2} \cdots a_{k_2,2} \cdots a_{1,h} \cdots a_{k_h,h}$. That is, one simply removes the parentheses in the cycle notation for ω to get another element of S_n written in permutation notation.

In the case that ω is a fixed-point free involution, so that ω has composition type $(2, 2, \ldots, 2)$, Can, Cherniavsky, and Twelbeck have shown [3, Proposition 7] that the map Ω above is exactly the map ϕ of Deodhar and Srinivasan which we generalize above. Furthermore, Can and Cherniavsky [2] have shown that if ω varies over all permutations of some fixed composition type, then the map Ω has image a graded sub-poset of S_n with respect to the Bruhat order, and they obtain generalizations of many results we obtain in this paper. In particular, in the case that ω has composition type $(m, m, \ldots, m)$ as in this paper, the map Ω is exactly $\Omega(\omega) = \phi(\omega)^{-1}$. We refer the reader to the paper [2] for more details.

4. EL-labelings and EL-shellability

Let $(P, \leq)$ be a finite graded poset, and let $\mathrm{cov}(P) = \{(x, y) \in P \times P \mid y \text{ covers } x\}$ be the set of edges of the Hasse diagram for P. An *edge labeling* of P is a function $\lambda : \mathrm{cov}(P) \to \Lambda$, where Λ is another poset. If $x_0 < x_1 < \cdots < x_n$ is an unrefinable chain c in P, then we extend λ to label c by $\lambda(c) = (\lambda(x_0, x_1), \lambda(x_1, x_2), \ldots, \lambda(x_{n-1}, x_n))$. The chain c is then called *rising* if $\lambda(x_0, x_1) \leq \lambda(x_1, x_2) \leq \cdots \leq \lambda(x_{n-1}, x_n)$ in Λ. The edge labeling λ is then called an *EL-labeling* if for every $x, y \in P$ with $x < y$, there is a unique rising, unrefinable chain $c_{x,y}$ from x to y, and if c^* is any other unrefinable chain from x to y different from $c_{x,y}$, then $\lambda(c_{x,y})$ precedes $\lambda(c^*)$ in the lexicographical order. If the poset $(P, \leq)$ admits an EL-labeling, then we say that it is *EL-shellable*.

Consider S_n with the (strong) Bruhat order introduced in Section 3. Define $\Lambda = \{(i, j) \in [n] \times [n] \mid i < j\}$ and order Λ lexicographically. Define an edge labeling $\lambda : \mathrm{cov}(S_n) \to \Lambda$ by $\lambda(\pi, \sigma) = (i, j)$, where i and j are the elements interchanged to obtain σ from π. Then λ is an EL-labeling of S_n [7]. This will be the EL-labeling of the symmetric group with the Bruhat order to which we shall refer for the rest of this section.

Now consider the poset $E(n_m)$, defined in Section 3 as a sub-poset of $S(n_m)$ with the Bruhat order. We see next that if we restrict the EL-labeling λ of $S(n_m)$ to $E(n_m)$, then this results in an EL-labeling of $E(n_m)$. This result and its proof are completely analogous to [5, Proposition 3.2].

Proposition 4.1. *The poset $E(n_m)$ is an EL-shellable graded poset, with grading and EL-labeling obtained by restriction from $S(n_m)$ under the Bruhat order.*

Proof. By [5, Proposition 3.1], it is enough to show that $E(n_m)$ contains a maximal element under the Bruhat order, $E(n_m)$ contains the minimal element of $S(n_m)$, and for all $\pi, \rho \in E(n_m)$ with $\pi < \rho$, the unique rising unrefinable chain $c_{\pi,\rho}$ in $S(n_m)$ lies completely in $E(n_m)$.

Like in the proof of Proposition 3.2, we will identify the linearly ordered set $[n_m]$ with $\{1, 2, \ldots, mn\}$, when they appear as indices in $\pi \in S(n_m)$, so $\pi = \pi_{1_1}\pi_{1_2} \cdots \pi_{n_m}$ will be written as $\pi = \pi_1\pi_2 \cdots \pi_{mn}$.

First, the element $1_1 1_2 \cdots 1_m 2_1 \cdots 2_m \cdots n_1 \cdots n_m$ is both the minimal element of $S(n_m)$ and an element of $E(n_m)$. Next, consider the element

$$\zeta = 1_1 2_1 \cdots n_1 n_m n_{m-1} \cdots n_2 (n-1)_m (n-1)_{m-1} \cdots (n-1)_2 \cdots 1_m 1_{m-1} \cdots 1_2 \in E(n_m).$$

We claim that ζ is a maximal element of $E(n_m)$. Let $\pi = \pi_1 \cdots \pi_{nm} = \pi_{1_1}\pi_{1_2} \cdots \pi_{n_m} \in E(n_m)$. If $\pi_1 \cdots \pi_n \neq 1_1 2_1 \cdots n_1$, find the least $i \geq 2$ such that $\pi_1 \cdots \pi_{i-1} = 1_1 \cdots (i-1)_1$, and then $i_1 = \pi_l$, for some $l > i$. Since $\pi \in E(n_m)$, then we must have $\pi_i, \pi_{i-1}, \ldots, \pi_{l-1} \in \{1_2, \ldots, 1_m, 2_2, \ldots, 2_m, \ldots, (i-1)_2, \ldots, (i-1)_m\}$. We may then make a sequence of inversion increasing interchanges, first π_l with π_{l-1}, then π_l with π_{l-2}, until we have obtained $\pi_1 \cdots \pi_{i-1}\pi_l\pi_i\pi_{i+1} \cdots \pi_{l-1}\pi_{l+1} \cdots \pi_{nm} = 1_1 \cdots (i-1)_1 i_1 \pi_i \cdots \pi_{nm}$. By induction, we may obtain from π a permutation σ of the form $\sigma = 1_1 2_1 \cdots n_1 \sigma_{n+1} \cdots \sigma_{nm}$ by a sequence of inversion increasing interchanges, so that $\pi \leq \sigma$. Note that any such σ in $S(n_m)$ is also an element of $E(n_m)$. It follows that we must have $\sigma \leq \zeta$, since $n_m > n_{m-1} > \cdots > n_2 > \cdots > 1_2$, and $n_m n_{m-1} \cdots n_2 \cdots 1_2$ corresponds to the maximal element of $S(n_{m-1})$ (shifting each i_j to i_{j-1}). Thus ζ is the maximal element of $E(n_m)$.

Now let $\pi, \rho \in E(n_m)$ such that $\pi < \rho$, and consider the unique unrefinable rising chain $c_{\pi,\rho}$ from π to ρ in $S(n_m)$. Let $l_j \in [n_m]$ be the least element such that $\pi^{-1}(l_j) \neq \rho^{-1}(l_j)$. Then [7, Remark 2] $\pi^{-1}(l_j) < \rho^{-1}(l_j)$. Suppose that $l_j \in \{1_1, 2_1, \ldots, n_1\}$. If $\pi^{-1}(l_j) = s$, then $\rho^{-1}(l_j) > s$, while for every $k_i < l_j$, $\pi^{-1}(k_i) = \rho^{-1}(k_i)$. Then we must have $\rho_s > l_j$. But now, $l_j \in \{1_1, 2_1, \ldots, n_1\}$, $\rho_s > l_j$, and ρ_s appears to the left of l_j in ρ. But this contradicts $\rho \in E(n_m)$. Thus, we must have $l_j \in \{1_2, \ldots, 1_m, 2_2, \ldots, 2_m, \ldots, n_2, \ldots, n_m\}$.

Now let $t_i \in [n_m]$ be the least element such that $t_i > l_j$ and $\pi^{-1}(l_j) < \pi^{-1}(t_i) \leq \rho^{-1}(l_j)$. Now write $\pi = \alpha_1 l_j \alpha_2 t_i \alpha_3$, where $\alpha_1, \alpha_2, \alpha_3$ are strings of elements from $[n_m]$. Consider the element $\omega = \alpha_1 t_i \alpha_2 l_j \alpha_3 \in S(n_m)$ obtained by exchanging l_j and t_i in π. Then by [7, Remark 5], ω is the element immediately after π in the unique rising chain $c_{\pi,\rho}$. We claim $\omega \in E(n_m)$, which will be enough to see that $c_{\pi,\rho}$ is contained in $E(n_m)$, by induction. Suppose $i \geq 2$, so $t_i \notin \{1_1, 2_1, \ldots, n_1\}$. Then $l_j < t_i$ and $j \geq 2$, so $l_j < t_1 < t_i$. Thus t_1 is not in the string α_2 by how we have chosen t_i. This implies $\omega = \alpha_1 t_i \alpha_2 l_j \alpha_3 \in E(n_m)$. Next suppose $i = 1$, so $t_i = t_1$. Then $l_j < (l+1)_1 < (l+2)_1 < \cdots < (t-1)_1 < t_1$. Thus none of $(l+1)_1, (l+2)_1, \ldots, (t-1)_1$ are in the string α_2. Since $\pi = \alpha_1 l_j \alpha_2 t_i \alpha_3 \in E(n_m)$, then l_1 is in the string α_1, and t_h for $h \geq 2$ are all in the string α_3. Thus $\omega = \alpha_1 t_i \alpha_2 l_j \alpha_3 \in E(n_m)$ again. So $c_{\pi,\rho}$ is contained in $E(n_m)$ as claimed. □

We now obtain our first main result of this section.

Theorem 4.1. *$(F^{(m)}(mn), \preceq)$ is an EL-shellable graded poset, of rank $\frac{(m-1)n(mn-2)}{2}$, with grading given by wt_m, and with rank generating function given by $\frac{[mn]_q!}{[mn]_q[m(n-1)]_q\cdots[m]_q}$.*

Proof. Since $E(n_m)$ is a graded EL-shellable poset by Proposition 4.1, and $F^{(m)}(mn)$ is isomorphic to $E(n_m)$ as a poset by Proposition 3.2, then $F^{(m)}(mn)$ is a graded EL-shellable poset. Since the order isomorphism ϕ maps the weight function wt_m of $F^{(m)}(mn)$ to the number of inversions ι of an element of $E(n_m)$ by Proposition 3.1, which is the grading for $E(n_m)$ under the Bruhat order, then wt_m provides a grading for $F^{(m)}(mn)$ under the partial order $\preceq$. Finally, the rank generating function is then given by

$$\sum_{\delta\in F^{(m)}(mn)} q^{\mathrm{wt}_m(\delta)} = \frac{[mn]_q!}{[mn]_q[m(n-1)]_q\cdots[m]_q},$$

by Corollary 2.1, and one can compute directly that the degree of this polynomial is $\frac{(m-1)n(mn-2)}{2}$, which is thus the rank of the graded poset $(F^{(m)}(mn), \preceq)$. □

We now give some notation in order to state and prove our last result. Let P be a finite graded poset with minimal element $\hat{0}$ and maximal element $\hat{1}$, and let μ_P be the Möbius function for P. Define $\overline{P} = P \setminus \{\hat{0}, \hat{1}\}$, and let $\Delta(\overline{P})$ be the order complex of $\overline{P}$. That is, $\Delta(\overline{P})$ is the simplicial complex with faces given by chains in $\overline{P}$, where a chain c consisting of n elements gives a face of dimension $n-1$. So, if the graded poset P has rank d, then $\Delta(\overline{P})$ has dimension $d-2$. We let $|\Delta(\overline{P})|$ denote the topological space constructed from the complex $\Delta(\overline{P})$ (see [10, Section 3.8]), and then $\Delta(\overline{P})$ triangulates the space $|\Delta(\overline{P})|$. When P is a finite graded poset which admits an EL-labeling λ, then the complex $\Delta(\overline{P})$ is *shellable* [1], which is why P is then called EL-shellable. We do not define the notion of a shellable complex here, but it can be found in [4], for example.

We now need a lemma. The minimal and maximal elements $\hat{0}$ and $\hat{1}$ in $E(n_m)$ are $\hat{0} = 1_1\cdots 1_m 2_1\cdots 2_m\cdots n_1\cdots n_m$ and $\hat{1} = 1_1 2_1\cdots n_1 n_m n_{m-1}\cdots n_2 (n-1)_m\cdots(n-1)_2\cdots 1_m\cdots 1_2$, which we showed in Proposition 4.1. The following result and its proof are adapted exactly from [5, p. 197, Proof of Theorem 1.3].

Lemma 4.1. *In the EL-shellable graded poset $E(n_m)$ with EL-labeling λ, there is no unrefinable chain c from $\hat{0}$ to $\hat{1}$ with a descent at every level (unless $n = m = 2$). That is, there is no unrefinable c, say $\hat{0} = x_0 < x_1 < x_2 < \cdots < x_k = \hat{1}$ with edge labels satisfying $\lambda(x_0, x_1) > \lambda(x_1, x_2) > \cdots > \lambda(x_{k-1}, x_k)$.*

Proof. Suppose such an unrefinable chain does exist. The smallest entry in $\hat{0}$ which moves at some point in the chain is 1_2, and since the edge labels are descending in the lexicographical ordering, the last edge labels all must be of the form $(1_2, b)$ for some $b \in [n_m]$, and no other edge labels earlier in the chain can be of this form. This implies that one element in the chain must be the permutation

$$\pi = 1_1 1_2 2_1 3_1\cdots n_1 n_m n_{m-1}\cdots n_2\cdots 1_m 1_{m-1}\cdots 1_3.$$

In the subchain of c from π to $\hat{1}$, we then must have the edge labeled $(1_2, 2_1)$ occur before $(1_2, 2_2)$. However, if $m > 2$ or $n > 2$, there must be other edges in the chain between these. This implies that there will not be a descent at some point in this chain. □

We may now give our last main result, which is a direct generalization of [5, Theorem 1.3(ii)], and the proof we give is essentially identical.

Theorem 4.2. *The complex $\Delta(\overline{F^{(m)}(mn)})$ triangulates a ball of dimension $\frac{(m-1)n(mn-2)}{2} - 2$.*

Proof. We may equivalently prove the statement for $E(n_m)$ in place of $F^{(m)}(mn)$, since these are isomorphic as EL-shellable graded posets. Let $d = \frac{(m-1)n(mn-2)}{2}$. Consider a chain c in $\overline{E(n_m)}$ of length one less than maximal, so that such a chain is of the form $x_1 < \cdots < x_{i-1} < x_{i+1} < \cdots < x_{d-1}$ for some i, where x_{i+1} does not cover x_{i-1} in $E(n_m)$. It is known that the symmetric group under the Bruhat order is Eulerian [11], meaning that any rank 2 interval of $S(n_m)$ contains exactly two elements apart from its endpoints. Thus, the chain c is contained in at most 2 chains of maximal length in $\overline{E(n_m)}$, since the elements x_{i-1} and x_{i+1} have only two elements between them in $S(n_m)$. By [4, Proposition 1.2], it follows that $\Delta(\overline{E(n_m)})$ triangulates either a ball or a sphere of dimension $d-2$.

Now, by [10, Equation (3.54) and Theorem 3.14.2] and Lemma 4.1, it follows that $\mu_{E(n_m)}(\hat{0}, \hat{1}) = 0$. For a simplicial complex Δ, let $\tilde{\chi}(\Delta)$ denote its reduced Euler characteristic. By [10, Proposition 3.8.6], we have $\mu_{E(n_m)}(\hat{0}, \hat{1}) = \tilde{\chi}(\Delta(\overline{E(n_m)}))$, and so $\tilde{\chi}(\Delta(\overline{E(n_m)})) = 0$. Since the reduced Euler characteristic of a sphere is ± 1, while the reduced Euler characteristic of a ball is 0, we must have that $\Delta(\overline{E(n_m)})$ triangulates a ball of dimension $d-2$. □

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