

# STRONG REALITY PROPERTIES OF NORMALIZERS OF PARABOLIC SUBGROUPS IN FINITE COXETER GROUPS

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ABSTRACT. Let  $W$  be a finite Coxeter group,  $P$  a parabolic subgroup of  $W$ , and  $N_W(P)$  the normalizer of  $P$  in  $W$ . We prove that every element in  $N_W(P)$  is strongly real in  $N_W(P)$ , and that every irreducible complex character of  $N_W(P)$  has Frobenius-Schur indicator 1.

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*Dedicated to the memory of Larry Grove*

## 1. INTRODUCTION

Carter [2] gave a unified approach to describing the conjugacy classes of Weyl groups. One of the interesting results which follows from Carter's study is the fact that every element in a Weyl group  $W$  is the product of two involutions in  $W$  [2, Theorem C]. Here, an involution is taken to mean an element which squares to the identity, and so includes the identity itself as an involution. The statement that any element is the product of two involutions also holds true for the non-crystallographic irreducible finite Coxeter groups, and is thus true for all finite Coxeter groups. We will call an element of a finite group  $G$  strongly real in  $G$  if it is the product of two involutions in  $G$ .

Another property of the finite Coxeter groups is that all of their irreducible complex representations may be realized over the real numbers. This follows from the construction of the irreducible complex characters of the finite reflection groups, starting with Frobenius [4] in 1900 with the symmetric groups, and finalized by Grove [8] in 1974 with the non-crystallographic group of order 14400. In other words, every irreducible complex character of a finite Coxeter group has Frobenius-Schur indicator equal to 1.

Now let  $W$  be a finite Coxeter group and  $P$  a parabolic subgroup of  $W$ . The structure of the normalizers  $N_W(P)$  of  $P$  in  $W$  was studied by Lusztig [17, Section 5] and Howlett [11]. These groups arise naturally and are important in the representation theory of finite groups of Lie type, see [3, Section 9.2] for example. Although the groups of the form  $N_W(P)$  are not, in general, themselves Coxeter groups, there are several properties of Coxeter groups which the normalizers of parabolic subgroups also enjoy, see [10], for

instance. The main result of this paper is that the reality properties which finite Coxeter groups have, as described above, also hold for the normalizers of parabolic subgroups of finite Coxeter groups. In particular, our main theorem is:

**Theorem 1.1.** *Let  $W$  be a finite Coxeter group,  $P$  a parabolic subgroup of  $W$ , and let  $N_W(P)$  be the normalizer of  $P$  in  $W$ . Then the following hold.*

- (1) *Every element of  $N_W(P)$  is strongly real in  $N_W(P)$ .*
- (2) *Every irreducible complex character of  $N_W(P)$  has Frobenius-Schur indicator 1.*

Our method of proof for Theorem 1.1 is as follows. For each of the infinite families of irreducible finite Coxeter groups, we give a group-theoretic description of the normalizers of parabolic subgroups, and prove the statements in these cases based on this structure. For the finite list of exceptional Coxeter groups, we use the computer algebra system `GAP` to check the statements.

There are several results related to Theorem 1.1 which have been proved previously. Grove and Wang [19] showed that the Frobenius-Schur indicator of an irreducible complex character of a Sylow 2-subgroup of any finite Coxeter group is always 1. Guralnick and Montgomery [9, Theorem 1.3(1)] have shown that if  $E$  is an elementary abelian 2-subgroup of a finite Coxeter group  $W$ , then any element in the centralizer  $C_W(E)$  of  $E$  in  $W$  is the product of two involutions in  $C_W(E)$ , and the Frobenius-Schur indicator of any irreducible complex character of  $C_W(E)$  is 1. This result intersects with Theorem 1.1, in that centralizers of involutions of finite Coxeter groups are examples of normalizers of parabolic subgroups, or conjugates of normalizers of parabolic subgroups, see [15, Sec. 27-3].

It is well-known that for any finite group  $G$ , the number of irreducible complex characters that are real-valued is equal to the number of real conjugacy classes, that is, conjugacy classes of elements which are conjugate to their inverse. Less is known, however, about the connection between the irreducible characters with Frobenius-Schur indicator 1 and strongly real conjugacy classes. There are results given by Gow [6] which give some connections between the characters with Frobenius-Schur indicator 1 and strongly real classes of a finite group, although these are not necessarily equal in number. Theorem 1.1 provides another family of examples for which the number of strongly real classes is equal to the number of irreducible characters with Frobenius-Schur indicator 1.

This paper is organized as follows. In Section 2, we give definitions and results for strongly real classes, Frobenius-Schur indicators, wreath products, and Coxeter groups. In Section 3, we prove Theorem 1.1 for each of the infinite families of irreducible finite Coxeter groups, and in Section 4 we discuss the `GAP` code used to prove Theorem 1.1 for the exceptional Coxeter groups.

## 2. PRELIMINARY RESULTS

**2.1. Strong reality and Frobenius-Schur indicators.** Let  $G$  be a finite group. An element  $g \in G$  is called *real* if  $g$  is conjugate to  $g^{-1}$  in  $G$ . It may be checked that if  $g$  is real, then any element conjugate to  $g$  is also real, and we thus call the conjugacy class of  $g$  a real conjugacy class. We define an *involution* of  $G$  to be an element  $t \in G$  such that  $t^2 = 1$ , and in particular, we adopt the convention of [2, Theorem C] that the identity itself is an involution. The element  $g \in G$  is called *strongly real* if  $g$  is conjugate to  $g^{-1}$  in  $G$  by an involution of  $G$ . With our definition of involution, we obtain the convenient fact that  $g$  is a strongly real element of  $G$  if and only if  $g$  is a product of two involutions in  $G$ . Again, it may be checked that if  $g \in G$  is strongly real, then any element conjugate to  $g$  is also strongly real, and so the conjugacy class of  $g$  is called a strongly real conjugacy class of  $G$ . It follows immediately that the property that all elements in a group are strongly real is preserved under taking direct products, and conversely, if the property holds for a direct product then it holds for each factor. Since we apply this fact throughout, we state this as a lemma.

**Lemma 2.1.** *Let  $G_1, \dots, G_m$  be finite groups. Then every element in the direct product  $G_1 \times \dots \times G_m$  is strongly real if and only if every element in each  $G_i$  is strongly real.*

Given a finite group  $G$ , let  $(\rho, V)$  be a finite-dimensional, complex representation of  $G$ , where  $\rho : G \rightarrow \mathrm{GL}(V)$ . Let  $\chi = \chi_\rho$  be the character of  $(\rho, V)$ , and given a basis  $\mathcal{B}$  of the vector space  $V$ , let  $[\rho]_{\mathcal{B}} : G \rightarrow \mathrm{GL}(n, \mathbb{C})$  denote the matrix representation of  $G$  obtained from  $(\rho, V)$  by using the basis  $\mathcal{B}$ . The representation  $(\rho, V)$  of  $G$  is called a *real representation* if there exists a basis  $\mathcal{B}$  of  $V$  such that the image of  $[\rho]_{\mathcal{B}}$  is completely contained in  $\mathrm{GL}(n, \mathbb{R})$ . If  $(\rho, V)$  is a real representation, then its character  $\chi$  is real-valued, although the converse is not necessarily true. In the case that  $(\rho, V)$  is an irreducible representation of  $G$ , the *Frobenius-Schur indicator* of  $\rho$ , or of  $\chi$ , denoted  $\varepsilon(\rho)$  or  $\varepsilon(\chi)$ , tells us when  $(\rho, V)$  is a real representation. In particular, we define

$$\varepsilon(\chi) = \begin{cases} 1 & \text{if } (\rho, V) \text{ is a real representation,} \\ -1 & \text{if } \chi \text{ is real-valued, but } (\rho, V) \text{ is not real,} \\ 0 & \text{if } \chi \text{ is not real-valued.} \end{cases}$$

In this paper, we are concerned with finite groups such that all of their irreducible characters have Frobenius-Schur indicator equal to 1. As with the property of having all strongly real elements, the property of having all real irreducible representations is also preserved under direct products, as we prove below.

**Lemma 2.2.** *Let  $G_1, \dots, G_m$  be finite groups with the property that every complex irreducible character of each  $G_i$  has Frobenius-Schur indicator 1. Then every complex irreducible character of the direct product  $G_1 \times \dots \times G_m$  also has Frobenius-Schur indicator 1.*

*Proof.* Any complex irreducible character  $\chi$  of  $G = G_1 \times \cdots \times G_m$  is a product  $\chi = \chi_1 \times \cdots \times \chi_m$ , where  $\chi_i$  is a complex irreducible character of  $G_i$ , and if  $g = (g_1, \dots, g_m) \in G$ , then  $\chi(g) = \prod_{i=1}^m \chi_i(g_i)$  [13, Theorem 4.21]. Since we are assuming that  $\chi_i$  is afforded by a real representation  $\pi_i$ , then the character  $\chi$  is also afforded by the real representation  $\pi_1 \oplus \cdots \oplus \pi_m$ , and so  $\varepsilon(\chi) = 1$ .  $\square$

The next result follows immediately from [9, Lemma 3.2].

**Lemma 2.3.** *Let  $G$  be a finite group such that all of its complex irreducible characters have Frobenius-Schur indicator 1. If  $H$  is an index 2 subgroup of  $G$  such that all of the complex irreducible characters of  $H$  are real-valued, then every complex irreducible character of  $H$  also has Frobenius-Schur indicator 1.*

We note that Lemma 2.3 also holds for normal subgroups  $H$  of prime index, by applying [13, Corollary 6.19], although we only need the statement for index 2 subgroups.

As we mentioned in the introduction, the number of real conjugacy classes of a finite group  $G$  is equal to the number of real-valued irreducible characters of  $G$ , see [13, Problem 6.13]. So, the hypothesis in Lemma 2.3 that all of the irreducible characters of  $H$  are real-valued is satisfied if all elements of  $H$  are real (or strongly real).

**2.2. Wreath products.** Let  $G$  be a finite group, and let  $n$  be a positive integer,  $n \geq 2$ . Let  $G^n$  denote the direct product of  $n$  copies of  $G$ , and let  $S_n$  act on  $G^n$  by

$$(2.1) \quad \pi \circ (x_1, \dots, x_n) = (x_{\pi^{-1}(1)}, \dots, x_{\pi^{-1}(n)}),$$

where  $\pi \in S_n$  and  $x_i \in G$ . The *wreath product* of  $G$  with  $S_n$ , denoted  $G \wr S_n$ , is defined as the semidirect product  $G^n \rtimes S_n$ , where  $S_n$  acts on  $G^n$  as in (2.1). A typical element in  $G \wr S_n$  will be denoted

$$[x_1, \dots, x_n; \pi],$$

where  $(x_1, \dots, x_n) \in G^n$  and  $\pi \in S_n$ . The inverse of an element in  $G \wr S_n$  is given by

$$(2.2) \quad [x_1, \dots, x_n; \pi]^{-1} = [\pi^{-1} \circ (x_1^{-1}, \dots, x_n^{-1}); \pi^{-1}].$$

The conjugacy classes of  $G \wr S_n$  may be described explicitly, although all that we need is a specific representative element from each conjugacy class. The next result may be extracted from [14, pgs. 138-142].

**Lemma 2.4.** *Let  $G$  be a finite group with identity element  $1_G$ . Each conjugacy class of  $G \wr S_n$  contains an element of the form*

$$g = [x_1, 1_G, \dots, 1_G, x_{j_1+1}, 1_G, \dots, 1_G, x_{j_{k-1}+1}, 1_G, \dots, 1_G; \pi_1 \pi_2 \cdots \pi_k],$$

where  $\pi_1 = (1 \cdots j_1), \pi_2 = (j_1 + 1 \cdots j_2), \dots, \pi_k = (j_{k-1} + 1 \cdots n)$ , are disjoint cycles. Further, we may write  $g$  as a product of commuting elements,  $g = g_1 g_2 \cdots g_k$ , where

$$\begin{aligned} g_1 &= [x_1, 1_G, \dots, 1_G; \pi_1], \\ g_2 &= [1_G, \dots, 1_G, x_{j_1+1}, 1_G, \dots, 1_G; \pi_2], \\ &\vdots \\ g_k &= [1_G, \dots, 1_G, x_{j_{k-1}+1}, 1_G, \dots, 1_G; \pi_k]. \end{aligned}$$

We now generalize the statement [14, Lemma 4.2.13], which says that if  $G$  is a finite group such that every element is real, then every element of the wreath product  $G \wr S_n$  is real.

**Lemma 2.5.** *Let  $G$  be a finite group such that every element of  $G$  is strongly real. Then every element of  $G \wr S_n$  is strongly real.*

Furthermore, if  $G = \mathbb{Z}_2 \times K$ , then each element of  $G \wr S_n$  may be conjugated to its inverse by an involution conjugate to one of the form  $[x_1, \dots, x_n; \sigma]$ , where  $\sigma \in S_n$  and  $x_i = (0, k_i) \in \mathbb{Z}_2 \times K$ . In particular, if  $G = \mathbb{Z}_2$ , the conjugating involution can be chosen to be conjugate to one in (the complementary)  $S_n$ .

*Proof.* First, note that it suffices to prove the lemma for conjugacy class representatives. Fixing a conjugacy class in  $G \wr S_n$ , we choose a representative  $g = g_1 g_2 \cdots g_k$  as described in Lemma 2.4.

We write  $\{1, \dots, n\}$  as the disjoint union  $\{1, \dots, n\} = \bigcup_{i=1}^k B_i$  with

$$B_1 = \{1, \dots, j_1\}, B_2 = \{j_1 + 1, \dots, j_2\}, \dots, B_k = \{j_{k-1} + 1, \dots, n\},$$

$j_i$  as in Lemma 2.4. Define  $H_i$  to be the subgroup of  $G \wr S_n$  given by

$$H_i = \{[y_1, \dots, y_n; \pi] \in G \wr S_n : y_j = 1_G \text{ and } \pi \text{ fixes } j \text{ if } j \notin B_i\}.$$

Then  $H_i \cong G \wr S_{|B_i|}$  and the  $H_i$  form an internal direct product

$$H_1 \times H_2 \times \cdots \times H_k \leq G \wr S_n.$$

Note that the decomposition  $g = g_1 \cdots g_k$  is precisely the decomposition of  $g$  in the direct product  $H_1 \times \cdots \times H_k$ . Thus it suffices to show that for each  $g_i$ , there is an involution  $\tau_i \in H_i$  conjugating  $g_i$  to  $g_i^{-1}$ . We first consider the case  $i = 1$ .

Note that by (2.2),

$$g_1^{-1} = [1_G, \dots, 1_G, x_1^{-1}, 1_G, \dots, 1_G; \pi_1^{-1}]$$

with the  $x_1^{-1}$  in the  $j_1$ th position. Set  $x = x_1$  and choose an involution  $t \in G$  such that  $txt = x^{-1}$ .

We define a permutation,  $\text{rev} \in S_n$ , as follows:

$$\text{rev}(m) = j_1 - m + 1 \quad \text{for } m = 1, \dots, j_1,$$

and

$$\text{rev}(m) = m \quad \text{for } m = j_1 + 1, \dots, n.$$

Then  $|\text{rev}| = 2$ , so  $\text{rev}$  is an involution in  $S_n$  and conjugates  $\pi_1$  to its inverse, since

$$(\text{rev})\pi_1(\text{rev}) = \text{rev}(1 \cdots j_1)\text{rev} = (j_1 \cdots 1) = \pi_1^{-1}.$$

Let  $\tau = [t, \dots, t, 1_G, \dots, 1_G; \text{rev}] \in H_1$  with  $t$  in the  $1, \dots, j_1$  positions and  $1_G$  in positions  $j_1 + 1, \dots, n$ . Then  $\tau$  is an involution, since

$$\begin{aligned} \tau^2 &= [t, \dots, t, 1_G, \dots, 1_G; \text{rev}] \cdot [t, \dots, t, 1_G, \dots, 1_G; \text{rev}] \\ &= [t^2, \dots, t^2, 1_G, \dots, 1_G; \text{rev} \cdot \text{rev}] \\ &= [1_G, \dots, 1_G; (1)]. \end{aligned}$$

Moreover,  $\tau$  conjugates  $g_1$  to its inverse, since

$$\begin{aligned} \tau \cdot g_1 \cdot \tau &= \tau \cdot [x, 1_G \dots, 1_G; \pi_1] \cdot \tau \\ &= [t, \dots, t, tx, 1_G, \dots, 1_G; (\text{rev})\pi_1] \cdot [t, \dots, t, 1_G, \dots, 1_G; \text{rev}] \\ &= [t^2, \dots, t^2, txt, 1_G, \dots, 1_G; (\text{rev})\pi_1(\text{rev})] \\ &= [1_G, \dots, 1_G, x^{-1}, 1_G, \dots, 1_G; \pi_1^{-1}] = g_1^{-1}. \end{aligned}$$

Thus  $\tau_1 = \tau$  is the desired involution in  $H_1$ . In the same way, there are also involutions  $\tau_i \in H_i$  conjugating  $g_i$  to  $g_i^{-1}$  for  $i = 2, \dots, k$ . Thus  $\tau_1\tau_2 \cdots \tau_k$  is an involution in  $G \wr S_n$  which conjugates  $g$  to  $g^{-1}$ .

For the second statement, if  $G = \mathbb{Z}_2 \times K$ , it follows that we may choose  $t = (0, k)$  where  $k$  is an involution of  $K$ , and if  $G = \mathbb{Z}_2$ , this means  $\tau$  is an element of (the complementary)  $S_n$ .  $\square$

We also have a result analogous to Lemma 2.5 for the property that the Frobenius-Schur indicators of the characters of a group are all 1. The following is given in [14, Corollary 4.4.11].

**Lemma 2.6.** *Suppose  $G$  is a finite group such that all of its complex irreducible characters have Frobenius-Schur indicator 1. Then every complex irreducible character of  $G \wr S_n$  also has Frobenius-Schur indicator 1.*

**2.3. Coxeter groups and parabolic subgroups.** A *finite reflection group* is a finite group generated by reflections in  $\text{GL}(n, \mathbb{R})$ , for some fixed  $n$ . A *finite Coxeter group* is a finite group  $W$  generated by a set  $S$  of elements  $r_\alpha$ , subject to the defining relations  $r_\alpha^2 = 1$  for all  $r_\alpha \in S$ , and  $(r_\alpha r_\beta)^{m(\alpha, \beta)} = 1$  for all pairs of elements  $r_\alpha, r_\beta \in S$ , where  $m(\alpha, \beta)$  is a positive integer. As it turns out, the notions of a finite reflection group and a finite Coxeter group are equivalent, where the set  $S$  corresponds to a certain set of generating reflections, and so we may use both interpretations interchangeably, although we will find the abstract group interpretation more useful for our methods. Since we are only considering finite groups, we will often leave out the word “finite” in our discussions, so a “Coxeter group” should be taken to mean a finite Coxeter group throughout.

For a Coxeter group  $W$  with generating set  $S$  as above, we call elements of  $S$  the *fundamental reflections* or *fundamental generators* of  $W$ . The pair  $(W, S)$ , is called a *Coxeter system*. Corresponding to  $(W, S)$  is a *Coxeter*

*graph*, which has one vertex for each element of  $S$ , and an edge between the vertices corresponding to  $r_\alpha$  and  $r_\beta$  if  $m(\alpha, \beta) > 2$ , and this edge is labeled with the number  $m(\alpha, \beta)$ .

A finite Coxeter group  $W$  with Coxeter system  $(W, S)$  is called *irreducible* if its Coxeter graph is connected. The classification of finite Coxeter groups states that any finite Coxeter group is the direct product of irreducible Coxeter groups, and irreducible Coxeter groups are classified into certain types based on their Coxeter graphs. Given a type  $T$  of Coxeter group, we write the group of that type as  $W(T)$ . In particular, every irreducible finite Coxeter group is of one of the following types:

$$A_n(n \geq 1), B_n(n \geq 2), D_n(n \geq 4), E_6, E_7, E_8, F_4, H_3, H_4, I_2(m)(m \geq 5),$$

and we refer the reader to one of the standard references [7, 12, 15] for a more detailed discussion. The groups of type  $A$ ,  $B$ ,  $D$ ,  $E$ , and  $F$  are the *Weyl groups*, or *crystallographic* Coxeter groups, while the groups of type  $H$  and  $I$  are the *non-crystallographic* Coxeter groups, with the exception of  $I_2(6)$ , which is the same as the Weyl group of type  $G_2$ . The infinite families of irreducible Coxeter groups are  $W(A_n), W(B_n), W(D_n)$ , and  $W(I_2(m))$ , while the finite list of groups  $W(E_6), W(E_7), W(E_8), W(F_4), W(H_3)$ , and  $W(H_4)$  are the *exceptional* Coxeter groups.

If  $W$  is a finite Coxeter group with Coxeter system  $(W, S)$ , where  $S = \{r_\alpha \mid \alpha \in J\}$  for some indexing set  $J$ , a *parabolic subgroup* of  $W$ , or just a *parabolic* of  $W$ , is a subgroup  $P$  of the form  $P = \langle r_\gamma \mid \gamma \in K \rangle$ , for some subset  $K \subset J$ . In particular, a parabolic subgroup  $P$  is itself a Coxeter group, with fundamental generators given by  $\{r_\gamma \mid \gamma \in K\}$ . Consider the normalizer  $N_W(P)$  of a parabolic  $P$  in a Coxeter group  $W$ . Notice that if  $W$  is not an irreducible Coxeter group, then  $W = W_1 \times \cdots \times W_m$ , where each  $W_i$  is an irreducible Coxeter group, and a parabolic subgroup  $P$  of  $W$  is of the form  $P = P_1 \times \cdots \times P_m$ , where each  $P_i$  (possibly trivial) is a parabolic subgroup of  $W_i$ . It follows that the normalizer  $N_W(P)$  itself is a direct product,  $N_W(P) = N_{W_1}(P_1) \times \cdots \times N_{W_m}(P_m)$ . Thus, by Lemmas 2.1 and 2.2, in order to prove Theorem 1.1, we need only prove it for irreducible Coxeter groups.

In the next two sections, we prove Theorem 1.1 for each type of irreducible Coxeter group. The structure of the normalizers of parabolic subgroups of each type is studied in detail by Howlett [11]. While Howlett mainly describes these subgroups from a geometric standpoint, in Section 3 we describe them from the abstract group point of view.

### 3. INFINITE FAMILIES OF IRREDUCIBLE COXETER GROUPS

**3.1. Groups of type  $I$ .** The finite irreducible Coxeter group  $W = W(I_2(m))$ ,  $m \geq 5$ , is defined as  $W(I_2(m)) = \langle r_1, r_2 \mid r_1^2 = r_2^2 = 1, (r_1 r_2)^m = 1 \rangle$ . The only proper, non-trivial, parabolic subgroups of  $W(I_2(m))$  are  $\langle r_1 \rangle$  and  $\langle r_2 \rangle$ , whose normalizers are isomorphic by symmetry. So, we just consider

$P = \langle r_1 \rangle$ , and notice that  $N_W(P) = C_W(r_1)$ , where  $C_W(r_1)$  is the centralizer of  $r_1$  in  $W$ .

The centralizer  $C_W(r_1)$  depends on the parity of  $m$ , and a direct calculation gives that if  $m$  is odd, then  $C_W(r_1) \cong \mathbb{Z}_2$ , while if  $m$  is even, then  $C_W(r_1) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ . In either case, it is immediate that Theorem 1.1 holds for  $N_W(P) = C_W(r_1)$  when  $W = W(I_2(m))$ .

**3.2. Groups of type A.** The irreducible Coxeter group  $W = W(A_n)$ ,  $n \geq 1$ , is defined as

$$W(A_n) = \langle r_1, \dots, r_n \mid r_i^2 = 1, (r_i r_{i+1})^3 = 1, (r_i r_j)^2 = 1 \text{ for } j - i > 1 \rangle.$$

So,  $W(A_n)$  is generated by  $n$  involutions and is isomorphic to the symmetric group on  $n + 1$  letters, when  $r_i$  is identified with the transposition  $(i \ i + 1)$ .

To describe the parabolic subgroups of  $W(A_n) \cong S_{n+1}$ , we proceed as follows. By definition, a parabolic subgroup  $P$  is generated by a subset of the generators  $r_1, \dots, r_n$ , and so by selecting a subset  $\mathbf{p}$  of  $\mathbf{n} = \{1, \dots, n\}$  and  $P = \langle r_a \mid a \in \mathbf{p} \rangle$ . The subset  $\mathbf{p}$  can be written in a unique way as a disjoint union of intervals. Let  $t$  be the number of distinct lengths of the intervals appearing in the union and let  $k_i$  be the number of intervals of length  $n_i$  in the union. We extend this notation by defining  $n_0 = 0$  and letting  $k_0$  be the size of  $\mathbf{n} \setminus \mathbf{p}$ . Since we are only interested in  $P$  up to conjugacy in  $W(A_n)$ , we can assume that  $\mathbf{p}$  is then given as follows, where we abbreviate  $k_0 + \sum_{i=1}^j k_i n_i$  by  $I_j$ :

$$\mathbf{p} = \bigcup_{j=1}^t \bigcup_{i=1}^{k_j} [I_{j-1} + (i-1)n_j + 1, I_{j-1} + in_j].$$

Applying the identification of  $W(A_n)$  and  $S_{n+1}$  defined above it follows that  $P$  as a subgroup of  $S_{n+1}$  can be described as

$$P = (S_{n_0+1} \times \dots \times S_{n_0+1}) \times (S_{n_1+1} \times \dots \times S_{n_1+1}) \times \dots \times (S_{n_t+1} \times \dots \times S_{n_t+1}).$$

The structure of the normalizer of the subgroup  $P$  in  $S_{n+1}$  is well-known. For a proof of the following lemma, see [1], for example.

**Lemma 3.1.** *Suppose  $P$  is a parabolic subgroup of  $S_{n+1}$  given as above. Then*

$$N_{S_{n+1}}(P) \cong (S_{n_0+1} \wr S_{k_0}) \times \dots \times (S_{n_t+1} \wr S_{k_t}).$$

From Lemma 3.1, Lemma 2.5, and Lemma 2.1, it follows that part (1) of Theorem 1.1 holds for  $H = N_W(P)$  when  $W = W(A_n)$ . Part (2) of Theorem 1.1 follows for  $W = W(A_n)$  by Lemmas 3.1, 2.6, and 2.2.

**3.3. Groups of type B.** The Coxeter group  $W(B_n)$ ,  $n \geq 2$ , is defined as  $\langle r_1, \dots, r_n \mid r_i^2 = (r_1 r_2)^4 = 1, (r_j r_{j+1})^3 = 1 \text{ if } j > 1, (r_i r_j)^2 = 1 \text{ if } j - i > 1 \rangle$ . By mapping  $r_i$  to  $(i-1 \ i) \in S_n$  for  $1 < i \leq n$ , and  $r_1$  to  $(1, 0, \dots, 0) \in \mathbb{Z}_2^n$  (the elements of which we write additively), we obtain [7, pgs. 67-68]

$$W(B_n) \cong \mathbb{Z}_2 \wr S_n.$$

We will identify  $W(B_n)$  with  $\mathbb{Z}_2 \wr S_n$  through this isomorphism.

We may describe the parabolic subgroups of  $W(B_n) = \mathbb{Z}_2 \wr S_n$  as follows. If  $P = \langle X \rangle$  is a parabolic subgroup, first suppose that  $r_1 \notin X$ , so that  $P$  is a parabolic of  $\langle r_2, \dots, r_n \rangle = S_n$ . From Section 3.2, any parabolic of  $S_n = W(A_{n-1})$ , up to conjugation, can be written in the form

$$P = S_1^{k_1} \times S_2^{k_2} \times \cdots \times S_n^{k_n},$$

with the slight modification that we take  $k_i = 0$  in the case that no factor of  $S_i$  appears. If  $r_1 \in X$ , suppose that  $r_1, r_2, \dots, r_m \in X$ , but  $r_{m+1} \notin X$ . Then

$$P = W(B_m) \times P^*,$$

where  $W(B_m) = \langle r_1, \dots, r_m \rangle$ ,  $P^*$  is a parabolic of  $S_{n-m} = \langle r_{m+2}, \dots, r_n \rangle$ , and in the case  $m = 1$  we define the factor  $W(B_1) = \langle r_1 \rangle$ . If we now include the case  $m = 0$ , corresponding to  $r_1 \notin X$ , then we have that any parabolic  $P$  of  $\mathbb{Z}_2 \wr S_n$ , up to conjugation, is of the form

$$P = W(B_m) \times S_1^{k_1} \times S_2^{k_2} \times \cdots \times S_{n-m}^{k_{n-m}}$$

where  $m + \sum_{i=1}^n i \cdot k_i = n$ .

We begin with a general observation.

**Lemma 3.2.** *Let  $G$ ,  $S$ , and  $N$  be groups such that  $G = N \rtimes S$ . Let  $P$  be a subgroup of  $S$ , and define  $N^P$  to be the fixed point subgroup of  $N$  under the action of  $P$ . If we identify  $P$  with  $\{1_N\} \rtimes P$  as a subgroup of  $G$ , where  $1_N$  is the identity element of  $N$ , then  $N_G(P) = N^P \rtimes N_S(P)$ .*

*Proof.* Write an element of  $G = N \rtimes S$  as  $[x; \sigma]$  with  $x \in N$ ,  $\sigma \in S$ , and let  $\sigma \circ x$  denote the action of  $\sigma \in S$  on  $x \in N$ .

We first claim that  $N^P$  is normal in  $N_G(P)$ . Indeed, if  $x \in N^P$  and  $u \in N_G(P)$ , then it follows that  $uxu^{-1} \in N^P$ . Thus  $N^P \rtimes N_S(P) \leq N_G(P)$ .

Conversely, we claim that if  $[x; \sigma] \in N_G(P)$ , then  $x \in N^P$  and  $\sigma \in N_S(P)$ . For any  $[1_N; \pi] \in P \leq G$ , we have

$$[x; \sigma] \cdot [1_N; \pi] \cdot [x; \sigma]^{-1} = [x \cdot (\sigma\pi\sigma^{-1} \circ x^{-1}); \sigma\pi\sigma^{-1}] = [1_N; \pi'],$$

where  $\pi' = \sigma\pi\sigma^{-1} \in P$ . Thus  $\sigma \in N_S(P)$ , and

$$1_N = x \cdot (\sigma\pi\sigma^{-1} \circ x^{-1}) = x \cdot (\pi' \circ x^{-1}).$$

Then  $\pi' \circ x^{-1} = x^{-1}$ , and so  $x^{-1}$  is fixed under elements of  $P$  of the form  $\sigma\pi\sigma^{-1}$  for  $\sigma \in N_S(P)$  and  $\pi \in P$ , and thus by all elements of  $P$ . Therefore  $x^{-1} \in N^P$ , and so  $x \in N^P$  as claimed.  $\square$

We may now apply Lemma 3.2 to describe  $N_W(P)$ , where  $W = \mathbb{Z}_2 \wr S_n$ , and  $P$  is a parabolic such that  $P \leq S_n = \langle r_2, \dots, r_n \rangle$ .

**Lemma 3.3.** *Let  $P \subseteq \langle r_2, \dots, r_n \rangle$  be a parabolic of  $W = W(B_n)$  of the form*

$$P = S_1^{k_1} \times S_2^{k_2} \times \cdots \times S_n^{k_n}.$$

*Then*

$$N_W(P) = (\mathbb{Z}_2 \times S_1) \wr S_{k_1} \times (\mathbb{Z}_2 \times S_2) \wr S_{k_2} \times \cdots \times (\mathbb{Z}_2 \times S_n) \wr S_{k_n}.$$

*Proof.* Let  $N = \mathbb{Z}_2^n$ , where  $W = N \rtimes S_n$ . We know from Lemma 3.2 that  $N_W(P) = N^P \rtimes N_{S_n}(P)$  and from Lemma 3.1 that  $N_{S_n}(P) = (S_1 \wr S_{k_1}) \times \cdots \times (S_n \wr S_{k_n})$ . The elements of  $N^P$  are the elements of  $\mathbb{Z}_2^n$  fixed under the action of  $P = S_1^{k_1} \times \cdots \times S_n^{k_n}$ . These are exactly the elements of  $\mathbb{Z}_2^n$  such that, for each factor  $S_i$  in  $P$ , the entries in the corresponding  $i$  coordinates on which  $S_i$  acts non-trivially are constant. Then we may identify  $N^P$  with  $\mathbb{Z}_2^{k_1} \times \mathbb{Z}_2^{k_2} \times \cdots \times \mathbb{Z}_2^{k_n}$ , where the elements of the factor  $\mathbb{Z}_2^{k_i}$  correspond to the  $k_i$  sets of  $i$  coordinates on which  $S_i^{k_i}$  acts non-trivially. By Lemma 3.2,

$$N_W(P) = (\mathbb{Z}_2^{k_1} \times \mathbb{Z}_2^{k_2} \times \cdots \times \mathbb{Z}_2^{k_n}) \rtimes (S_1 \wr S_{k_1} \times \cdots \times S_n \wr S_{k_n}),$$

where the factor  $S_j \wr S_{k_j}$  acts non-trivially only on the  $\mathbb{Z}_2^{k_j}$  term. Thus

$$N_W(P) = (\mathbb{Z}_2^{k_1} \rtimes (S_1 \wr S_{k_1})) \times \cdots \times (\mathbb{Z}_2^{k_n} \rtimes (S_n \wr S_{k_n})).$$

In each factor  $\mathbb{Z}_2^{k_j} \rtimes (S_j \wr S_{k_j})$ ,  $S_j^{k_j}$  acts trivially on  $\mathbb{Z}_2^{k_j}$  by construction. Thus  $\mathbb{Z}_2^{k_j} \rtimes (S_j \wr S_{k_j}) = (\mathbb{Z}_2 \times S_j) \wr S_{k_j}$ , which gives the result.  $\square$

Finally, we deal with the case of a general parabolic subgroup of  $W(B_n)$ .

**Lemma 3.4.** *Let  $P$  be a parabolic of  $W = W(B_n)$  of the form*

$$P = W(B_m) \times S_1^{k_1} \times S_2^{k_2} \times \cdots \times S_{n-m}^{k_{n-m}}.$$

*Then*

$$N_W(P) = W(B_m) \times (\mathbb{Z}_2 \times S_1) \wr S_{k_1} \times (\mathbb{Z}_2 \times S_2) \wr S_{k_2} \times \cdots \times (\mathbb{Z}_2 \times S_{n-m}) \wr S_{k_{n-m}}.$$

*Proof.* Let  $P$  be a parabolic of the form described. Then we can view  $P$  as  $W(B_m) \times P^*$  where

$$P^* = S_1^{k_1} \times S_2^{k_2} \times \cdots \times S_{n-m}^{k_{n-m}}.$$

Let  $N = \mathbb{Z}_2^n$  where  $W(B_n) = N \rtimes S_n$ . Note that  $P \cap N = \mathbb{Z}_2^m$ , where  $W(B_m) = (P \cap N) \rtimes S_m$  and  $S_m = \langle r_2, \dots, r_m \rangle$ . Suppose that  $g \in N_W(P)$ . Then  $g \in N_W(P \cap N)$  since  $N$  is normal in  $W$ . Write  $g = [y_1, \dots, y_n; \sigma]$ , and define  $x = [x_1, \dots, x_n; (1)]$ , where  $x_i = 1$  for  $1 \leq i \leq m$ , and  $x_i = 0$  for  $i > m$ , so that  $x \in P \cap N$ . Then

$$\begin{aligned} gxg^{-1} &= [y_1, \dots, y_n; \sigma][x_1, \dots, x_n; (1)][y_{\sigma(1)}, \dots, y_{\sigma(n)}; \sigma^{-1}] \\ &= [x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)}; (1)]. \end{aligned}$$

Since  $gxg^{-1} \in P \cap N$ , we must have  $x_{\sigma^{-1}(i)} = 0$  for  $i > m$ , and thus  $\sigma$  permutes  $\{1, \dots, m\}$  disjointly from  $\{m+1, \dots, n\}$ . That is,  $\sigma \in S_m \times S_{n-m}$ , and thus  $g \in W(B_m) \times W(B_{n-m}) = \mathbb{Z}_2^m \rtimes (S_m \times S_{n-m})$ . Conversely, by essentially the same calculation as above, any element of  $W(B_m) \times W(B_{n-m})$  normalizes  $P \cap N$ , and thus  $N_W(P \cap N) = W(B_m) \times W(B_{n-m})$ .

We conclude that  $P = W(B_m) \times P^* \leq W(B_m) \times W(B_{n-m})$ , and thus

$$N_W(P) = N_{W(B_m)}(W(B_m)) \times N_{W(B_{n-m})}(P^*) = W(B_m) \times N_{W(B_{n-m})}(P^*).$$

Finally, the result follows by applying Lemma 3.3 to  $N_{W(B_{n-m})}(P^*)$ .  $\square$

By applying Lemma 3.4, we see that part (1) of Theorem 1.1 holds for  $W = W(B_n)$  by Lemmas 2.5 and 2.1, and part (2) of Theorem 1.1 follows for  $W = W(B_n)$  by Lemmas 2.6 and 2.2.

**3.4. Groups of type  $D$ .** The Coxeter group  $W = W(D_n)$  ( $n \geq 4$ ) is defined by  $W(D_n) = \langle s_1, \dots, s_n \mid s_i^2 = 1, (s_1 s_2)^2 = 1, (s_1 s_3)^3 = 1, (s_j s_{j+1})^3 = 1 \text{ if } j \geq 2, (s_i s_j)^2 = 1 \text{ if } j > i + 1 \text{ and } (i, j) \neq (1, 3) \rangle$ .

If we consider  $W(B_n)$  as in Section 3.3 with fundamental generators  $r_1, r_2, \dots, r_n$ , then we may consider  $W(D_n)$  as a subgroup of  $W(B_n)$  by letting  $s_1 = r_1 r_2 r_1$ , and  $s_i = r_i$  for  $i > 1$ , which may be seen by checking directly that  $r_1 r_2 r_1$  satisfies the relations which  $s_1$  does in the definition of  $W(D_n)$ . If we identify  $W(B_n)$  with  $\mathbb{Z}_2 \wr S_n$  as in Section 3.3, then  $W(D_n)$  is the index two subgroup of  $\mathbb{Z}_2 \wr S_n$  which consists of all elements which contain an even number of 1's in their  $\mathbb{Z}_2^n$ -coordinates. In this identification, we have  $r_1 r_2 r_1 = [1, 1, 0, \dots, 0; (1 \ 2)]$ . For the rest of this section, we will identify  $W(D_n)$  with this subgroup of  $W(B_n) = \mathbb{Z}_2 \wr S_n$ , and we will take the generators of  $W(D_n)$  to be  $s_1 = r_1 r_2 r_1$ ,  $s_i = r_i$  for  $i > 1$ .

By the result of Carter [2] mentioned in the introduction, we know that every element of  $W(D_n)$  is conjugate to its inverse by an involution in  $W(D_n)$ . We actually have the following stronger result, which we will need to apply later.

**Lemma 3.5.** *Consider  $W(D_n)$  as a subgroup of  $W(B_n) = \mathbb{Z}_2 \wr S_n$ . Every element of  $W(B_n)$  is conjugate to its inverse by an involution in  $W(D_n)$ .*

*Proof.* This follows immediately from the last statement of Lemma 2.5.  $\square$

Before analyzing the parabolic subgroups of  $W(D_n)$ , we first make the following basic group theoretical observations which will be useful to us.

**Lemma 3.6.** *Let  $G$  be a finite group, and  $H$  an index 2 subgroup of  $G$ . Let  $\tilde{P}$  be a subgroup of  $G$ , and let  $P = \tilde{P} \cap H$ . Then the following hold.*

- (i)  $N_H(P) = N_G(P) \cap H$ .
- (ii)  $[N_G(P) : N_H(P)] \leq 2$ .

Consider a parabolic subgroup  $P$  of  $W(D_n)$  with  $X$  the set of fundamental generators of  $P$ . If  $X \subseteq \{r_2, r_3, \dots, r_n\}$ , then  $P = \langle X \rangle$  is itself a parabolic subgroup of  $W(B_n)$ . If  $X \subseteq \{r_1 r_2 r_1, r_3, \dots, r_n\}$ , and  $r_1 r_2 r_1 \in X$ , then  $P^{r_1}$  is a parabolic subgroup of the first type, which is a parabolic of  $W(B_n)$ . It follows that we only need to consider one of these cases of parabolics of  $W(D_n)$ . That is, we only study the parabolic subgroups  $P = \langle X \rangle$  of  $W(D_n)$  such that either  $X \subseteq \{r_2, r_3, \dots, r_n\}$ , or  $r_2$  and  $r_1 r_2 r_1$  are both elements of  $X$ .

We first consider the case of a parabolic  $P = \langle X \rangle$  with  $X \subseteq \{r_2, \dots, r_n\}$ . We can already prove part (1) of Theorem 1.1 for this case.

**Lemma 3.7.** *Let  $P = \langle X \rangle$  be a parabolic subgroup of  $W = W(D_n)$  such that  $X \subseteq \{r_2, \dots, r_n\}$ . Then every element of  $N_W(P)$  is conjugate to its inverse by an involution in  $N_W(P)$ .*

*Proof.* Since  $P = \langle X \rangle$  with  $X \subseteq \{r_2, \dots, r_n\}$ ,  $P$  is also a parabolic subgroup of  $W(B_n)$ . From Lemma 3.3, we have

$$N_{W(B_n)}(P) = (\mathbb{Z}_2 \times S_1) \wr S_{k_1} \times (\mathbb{Z}_2 \times S_2) \wr S_{k_2} \times \cdots \times (\mathbb{Z}_2 \times S_n) \wr S_{k_n},$$

and by Lemma 3.6(i),  $N_W(P) = N_{W(B_n)}(P) \cap W$ . It suffices to show that every element in the factor  $(\mathbb{Z}_2 \times S_m) \wr S_{k_m}$ , for some  $m$ , is conjugate to its inverse by an element of the intersection of  $W(D_n)$  with the subgroup  $(\mathbb{Z}_2 \times S_m) \wr S_{k_m}$ . Let  $x \in (\mathbb{Z}_2 \times S_m) \wr S_{k_m}$ . By the second statement in Lemma 2.5,  $x$  is conjugate to its inverse by an involution conjugate to one of the form  $[t_1, \dots, t_{k_m}; \pi]$ , where  $t_i = (0, \sigma_i)$  and  $\sigma_i \in S_m$  is an involution. It follows that this involution is in the intersection of  $W(D_n)$  and  $(\mathbb{Z}_2 \times S_m) \wr S_{k_m}$ . Thus every element in  $N_W(P)$  is conjugate to its inverse by an involution in  $N_W(P)$ .  $\square$

We now concentrate on the case that  $P$  is a parabolic subgroup of  $W(D_n)$  which contains both of the generators  $r_1 r_2 r_1$  and  $r_2$ . In particular,  $P$  itself is not a parabolic subgroup of  $W(B_n)$ , but we can give the specific relationship between the normalizer of this parabolic in  $W(D_n)$ , and the normalizer of another parabolic in  $W(B_n)$ .

**Lemma 3.8.** *Let  $P$  be a parabolic subgroup of  $W = W(D_n)$  which contains both  $r_1 r_2 r_1$  and  $r_2$ . That is, we have  $P = \langle r_1 r_2 r_1, r_2, X \rangle$ , where  $X$  is some subset of  $\{r_3, \dots, r_n\}$ . If  $\tilde{P}$  is the parabolic subgroup of  $W(B_n)$  with  $\tilde{P} = \langle r_1, r_2, X \rangle$ , then*

$$N_{W(B_n)}(\tilde{P}) = N_{W(B_n)}(P).$$

*Proof.* Note that  $P = \tilde{P} \cap W$ , and  $P = W(D_m) \times P^*$ , where

$$P^* = S_1^{k_1} \times S_2^{k_2} \times \cdots \times S_{n-m}^{k_{n-m}}.$$

If we replace the parabolic  $P$  which appears in Lemma 3.3 by the parabolic  $P^*$  here, we deduce from the argument given in the proof of Lemma 3.3 that  $N_{W(B_n)}(\tilde{P}) = N_{W(B_n)}(P)$ , as desired.  $\square$

We are now prepared to finish the proof of part (1) of Theorem 1.1 for the case  $W = W(D_n)$ .

**Lemma 3.9.** *Let  $P$  be a parabolic subgroup of  $W = W(D_n)$  such that  $P = \langle r_1 r_2 r_1, r_2, X \rangle$  for some  $X \subseteq \{r_3, \dots, r_n\}$ . Then every element in  $N_W(P)$  is conjugate to its inverse by an involution in  $N_W(P)$ .*

*Proof.* Let  $\tilde{P} = \langle r_1, r_2, X \rangle$ , a parabolic in  $W(B_n)$ . By Lemma 3.8,  $N_{W(B_n)}(P) = N_{W(B_n)}(\tilde{P})$ , and by Lemma 3.6(i),  $N_W(P) = N_{W(B_n)}(\tilde{P}) \cap W$ . By Lemma 3.4, we have  $N_{W(B_n)}(\tilde{P}) = W(B_m) \times N^*$ , where

$$N^* = (\mathbb{Z}_2 \times S_1) \wr S_{k_1} \times \cdots \times (\mathbb{Z}_2 \times S_{n-m}) \wr S_{k_{n-m}}.$$

Now take  $x \in N_{W(B_n)}(\tilde{P})$ , with  $x = x_0 x^*$ , where  $x_0$  is in  $W(B_m)$ , and  $x^*$  is in  $N^*$ . By Lemma 3.7, there is an involution  $\tau^*$  in  $W(D_n)$ , also in the

factor  $N^*$ , which conjugates  $x^*$  to its inverse. By Lemma 3.5, there exists an involution  $\tau_0$ , in the subgroup  $W(D_m)$  of the factor  $W(B_m)$ , such that  $\tau_0$  is an involution and  $\tau_0$  conjugates  $x_0$  to its inverse. From the embedding of the  $W(B_m)$  factor in  $W(B_n)$ , it follows that  $\tau_0 \in W(D_n)$ . Now, taking  $\tau = \tau_0\tau^*$ , we have  $\tau$  is an involution in  $W(D_n) \cap N_{W(B_n)}(\tilde{P})$ , and  $\tau x \tau = x^{-1}$ . Thus, every element in  $N_W(P)$  is conjugate to its inverse by an involution of  $N_W(P)$ .  $\square$

Now that we have shown part (1) of Theorem 1.1 holds for  $W = W(D_n)$ , we know that in particular, all of the irreducible complex characters of  $N_W(P)$  are real-valued, for any parabolic subgroup  $P$  of  $W$ . By Lemmas 3.6(ii) and 3.8,  $N_W(P)$  is a subgroup of index at most 2 of the normalizer of a parabolic (either  $P$  itself or  $\tilde{P}$ ) of  $W(B_n)$ . Since part (2) of Theorem 1.1 holds for the normalizers of parabolic subgroups of  $W(B_n)$ , it follows from Lemma 2.3 that part (2) of Theorem 1.1 also holds for  $W = W(D_n)$ . This concludes the proof of Theorem 1.1 for the case that  $W$  is in one of the infinite families of irreducible Coxeter groups.

#### 4. EXCEPTIONAL IRREDUCIBLE COXETER GROUPS AND GAP CODE

To verify Theorem 1.1 for the exceptional Coxeter groups, we make use of the computer algebra system **GAP**, see [5]. Very conveniently, **GAP** offers procedures that construct any of the Weyl groups, in particular the exceptional Weyl groups. For the non-crystallographic Coxeter groups  $W(H_3)$  and  $W(H_4)$ , we take advantage of the fact that they actually arise as subgroups of  $W(E_8)$ , see [16].

In addition, we make use of **GAP**'s efficient algorithms to compute the character table of a finite group given by a set of generators. The reason for doing so is that given the character table we can easily derive whether all the irreducible characters have Frobenius-Schur indicator 1. Furthermore, we can decide whether an element that is not an involution in  $G$  is the product of two involutions, and hence strongly real. This is based upon the following observation, for a proof see for example [18, pg. 125].

**Lemma 4.1.** *Let  $G$  be a finite group and let  $C_1, C_2, C_3$  be conjugacy classes of elements of  $G$ . Then the number  $n_{123}$  of pairs  $(x, y)$  in  $C_1 \times C_2$  such that the product  $xy$  is a fixed element in  $C_3$  is given by the following formula*

$$n_{123} = \frac{|C_1||C_2|}{|G|} \sum_{\chi \in \text{Irr}(G)} \frac{\chi(C_1)\chi(C_2)\overline{\chi(C_3)}}{\chi(1)}.$$

To gain efficiency we present the exceptional Coxeter groups as permutation groups on the the roots of a fixed length, since permutation group algorithms are the most efficient algorithms for computing in a given finite group. As mentioned above, we consider  $W(H_3)$  and  $W(H_4)$  as subgroups of  $W(E_8)$ , and in particular as permutation groups on the roots of  $W(E_8)$ .

The overall approach for checking strongly real for an exceptional Coxeter group  $W$  is as follows.

- 1) Construct generators for  $W$  as permutations.
- 2) Construct the parabolic subgroups of  $W$ .
- 3) For each parabolic  $P$  of  $W$ 
  - 1) Compute the normalizer  $N := N_W(P)$  and the character table of  $N$ .
  - 2) For each conjugacy class  $C_3$  not consisting of involutions in  $N$  and all pairs of involution classes  $C_1, C_2$  in  $N$ : Compute the number  $n_{123}$  using the GAP-command `ClassMultCoefficientCharacterTable`.
  - 3) If there is any class  $C_3$  for which  $n_{123}$  is zero for all involution classes  $C_1, C_2$ , return false.
- 4) Return true.

To check the Frobenius-Schur indicators is actually easier since we only have to apply the GAP-function `Indicator` to the character table of  $N$  and check that the indicators are all 1.

The GAP procedures implementing the outline can be downloaded from <http://math.arizona.edu/~klux/stronglyrealCheckStronglyReal2.g> and <http://math.arizona.edu/~klux/CheckSchurIndicators2.g>.

To give an impression on the computing time spent for verifying Theorem 1.1: For the largest group to consider  $W(E_8)$ , it took 4730 seconds on a DELL desktop computer running Suse Linux 11.3 with 4 Gigabyte of RAM and an Intel(R) Core(TM)2 Duo CPU E6550 at 2.33GHz to verify that all elements of  $W(E_8)$  are strongly real. To confirm that all Frobenius-Schur indicators are 1 took 801 seconds.

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