manuscripta math. 132, 145–158 (2010) 

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# Shintani lifting and real-valued characters

Received: 21 September 2009
Published online: 3 February 2010

**Abstract.** We study Shintani lifting of real-valued irreducible characters of finite reductive groups. In particular, if $\mathbf{G}$ is a connected reductive group defined over $\mathbb{F}_q$, and $\psi$ is an irreducible character of $\mathbf{G}(\mathbb{F}_{q^m})$ which is the lift of an irreducible character $\chi$ of $\mathbf{G}(\mathbb{F}_q)$, we prove $\psi$ is real-valued if and only if $\chi$ is real-valued. In the case $m = 2$, we show that if $\chi$ is invariant under the twisting operator of $\mathbf{G}(\mathbb{F}_{q^2})$, and is a real-valued irreducible character in the image of lifting from $\mathbf{G}(\mathbb{F}_q)$, then $\chi$ must be an orthogonal character. We also study properties of the Frobenius–Schur indicator under Shintani lifting of regular, semisimple, and irreducible Deligne–Lusztig characters of finite reductive groups.

## 1. Introduction

Shintani [23] gave a bijection from the irreducible complex characters of the general linear group $\mathrm{GL}(n, \mathbb{F}_q)$ over a finite field $\mathbb{F}_q$, to the Frobenius-invariant complex irreducible characters of the general linear group $\mathrm{GL}(n, \mathbb{F}_{q^m})$ over an extension field $\mathbb{F}_{q^m}$, through an identity of character values. This correspondence has come to be known as *Shintani lifting*. The theory has been extended to other algebraic groups over a finite field by Kawanaka [13–15] and Gyoja [12], as well as understood in a more geometric context as *Shintani descent*, by Digne and Michel [7] and Shoji [24], among others. Shintani lifting and descent have played a key role in the development of the character theory of finite groups of Lie type, and has also influenced the development of the lifting theory and base change of automorphic forms and representations (see [15, Sect. 1.1]).

In this paper, we study Shintani lifting of real-valued irreducible complex characters of finite reductive groups. There are classical results on the complex characters of finite groups for which there are analogous theorems for real-valued characters of finite groups, such as the results on degrees of real-valued characters obtained by Navarro, Sanus, and Tiep [20]. Such results indicate that the theory of real-valued characters of finite groups inherits many of the rich aspects of the classical character theory. Because finite reductive groups are such an important class of finite groups, and Shintani lifting is a fundamental part of their character theory, it is a natural problem to investigate the behavior of real-valued characters

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*Mathematics Subject Classification (2000)*: 20C33

DOI: 10.1007/s00229-010-0339-3

under Shintani lifting. Furthermore, we expect that corresponding to our results for Shintani lifting of real-valued characters of finite reductive groups, there should be analogous results for the base change of self-dual representations of $p$-adic groups.

The main results and organization of this paper are as follows. In Sect. 2, we give results and definitions regarding norm maps and Shintani lifting for finite reductive groups. In Sect. 3, we obtain our first main result in Theorem 3.2, which states that for $\mathbf{G}$ a connected reductive group defined over $\mathbb{F}_q$, if an irreducible character $\chi$ of $\mathbf{G}(\mathbb{F}_{q^m})$ is the Shintani lift of the character $\psi$ of $\mathbf{G}(\mathbb{F}_q)$, then $\chi$ is real-valued if and only if $\psi$ is real-valued. We apply Theorem 3.2 to obtain a set of irreducible characters of the finite symplectic groups which have no even degree Shintani lifts (Corollary 3.1), and to obtain a bijection between irreducible real-valued characters of the finite general linear and unitary groups (Corollary 3.2).

In Sect. 4, we concentrate on quadratic lifting of characters, from $\mathbf{G}(\mathbb{F}_q)$ to $\mathbf{G}(\mathbb{F}_{q^2})$. After giving the necessary results on Frobenius–Schur indicators, we prove Theorem 4.2, which states that if $\chi$ is an irreducible character of $\mathbf{G}(\mathbb{F}_{q^2})$, which is both invariant under the twisting operator and the lift of a real-valued irreducible character of $\mathbf{G}(\mathbb{F}_q)$, then $\chi$ must be the character of a real representation.

Finally, in Sects. 5 and 6, we concentrate on real-valued characters which are regular or semisimple, and irreducible Deligne–Lusztig characters. We begin by showing that the central character of an irreducible character behaves well under Shintani lifting (Lemma 5.1), and we use this to describe how the Frobenius–Schur indicator of a real-valued semisimple or regular character is affected by Shintani lifting, in Theorem 5.2. In Sect. 6, we describe a bijection from the irreducible Deligne–Lusztig characters of $\mathbf{G}(\mathbb{F}_q)$ to the irreducible Frobenius-invariant Deligne–Lusztig characters of $\mathbf{G}(\mathbb{F}_{q^m})$ (where $\mathbf{G}$ is now assumed to have connected center), which is known in many cases to coincide with Shintani lifting, by results of Gyoja [12] and Digne [6]. We conclude with Theorem 6.1, which states that this map restricts to give a bijection between the subsets of real-valued irreducible Deligne–Lusztig characters of $\mathbf{G}(\mathbb{F}_q)$ and $\mathbf{G}(\mathbb{F}_{q^m})$, and this map has the expected effect on the Frobenius–Schur indicator.

## 2. Norm maps and Shintani lifting

Let $\mathbf{G}$ be a connected reductive group over $\bar{\mathbb{F}}_q$, defined over $\mathbb{F}_q$ with some Frobenius map $\sigma$. Let $m$ be a positive integer, and define $G = \mathbf{G}(\mathbb{F}_{q^m}) = \mathbf{G}^{\sigma^m}$ and $H = \mathbf{G}(\mathbb{F}_q) = \mathbf{G}^{\sigma}$. In particular, $G^{\sigma} = H$. Now, define $G\langle\sigma\rangle$ to be the split extension of $G$ by the cyclic group $\langle\sigma\rangle$ of order $m$. That is,

$$G\langle\sigma\rangle = \langle G, \sigma \mid \sigma^m = 1, \sigma^{-1} g \sigma = {}^{\sigma}g \quad \text{for all } g \in G\rangle.$$

By a slight abuse of notation, we are letting $\sigma$ denote both an automorphism of $G$, and an element in the group $G\langle\sigma\rangle$.

Two elements $x, y \in G$ are said to be $\sigma$*-conjugate* in $G$ if there is an element $g \in G$ such that ${}^{\sigma}g^{-1} x g = y$. As can be checked, two elements $x, y \in G$ are $\sigma$-conjugate in $G$ if and only if the elements $\sigma x$ and $\sigma y$ in the coset $\sigma G$ of $G\langle\sigma\rangle$

are conjugate in $G\langle\sigma\rangle$. That is, the set of $\sigma$-conjugacy classes in $G$ are in natural bijective correspondence with the set of $G\langle\sigma\rangle$ conjugacy classes in the coset $\sigma G$.

For any $x \in \mathbf{G}$, by the Lang–Steinberg Theorem, there exists an $\alpha_x \in \mathbf{G}$ such that ${}^{\sigma}\alpha_x^{-1}\alpha_x = x$. It follows that $x \in G$ if and only if the element $\alpha_x\,({}^{\sigma^m}\alpha_x^{-1})$ is in $H = G^{\sigma}$, where

$$\alpha_x\,({}^{\sigma^m}\alpha_x^{-1}) = \alpha_x \left(\prod_{i=1}^{m}({}^{\sigma^{m-i}}x)\right)\alpha_x^{-1}.$$

This allows us to define a map from the $\sigma$-conjugacy classes of $G$ (or $G\langle\sigma\rangle$ conjugacy classes in $\sigma G$) to the set of conjugacy classes in $H = G^{\sigma}$. This map is given in the following Proposition, which was proven by Shintani [23] in the case of $\mathrm{GL}(n, \mathbb{F}_q)$, and proven in the more general case by several authors, including Kawanaka [13], Gyoja [12], and Digne and Michel [7]. We call this map the *Shintani norm map* from $\mathbf{G}(\mathbb{F}_{q^m})$ to $\mathbf{G}(\mathbb{F}_q)$.

**Proposition 2.1 (Shintani norm map).** *Given a $\sigma$-conjugacy class $[x]_\sigma$ of $G = \mathbf{G}(\mathbb{F}_{q^m})$, and thus a $G\langle\sigma\rangle$ conjugacy class $[\sigma x]_{G\langle\sigma\rangle}$ in $\sigma G$, let $\alpha_x \in \mathbf{G}$ be such that ${}^{\sigma}\alpha_x^{-1}\alpha_x = x$. Define the map $N_{\mathbb{F}_{q^m}/\mathbb{F}_q}$ by*

$$N_{\mathbb{F}_{q^m}/\mathbb{F}_q}([x]_\sigma) = [\alpha_x\,({}^{\sigma^m}\alpha_x^{-1})]_H,$$

*where the right-hand side is a conjugacy class in $H = \mathbf{G}(\mathbb{F}_q)$. Then $N_{\mathbb{F}_{q^m}/\mathbb{F}_q}$ is a bijection from the set of $\sigma$-conjugacy classes of $G$ to the set of conjugacy classes of $H$, and is independent of the choice of $\alpha_x \in \mathbf{G}$.*

We may also consider the Shintani norm map from $\mathbf{G}(\mathbb{F}_q)$ to $\mathbf{G}(\mathbb{F}_q)$, where $N_{\mathbb{F}_q/\mathbb{F}_q}$ permutes the conjugacy classes of $\mathbf{G}(\mathbb{F}_q)$. In this case, we denote $N_{\mathbb{F}_q/\mathbb{F}_q}$ by $T_{\mathbb{F}_q}$, or simply by $T$ if the field $\mathbb{F}_q$ is understood. If $\eta$ is any complex-valued class function on $\mathbf{G}(\mathbb{F}_q)$, define the *twisting operator* on $\eta$, denoted by $T^*_{\mathbb{F}_q}$, or simply $T^*$, by

$$T^*(\eta) = \eta \circ T.$$

It is a result of Asai [1,2] and Digne and Michel [7] that if $\chi$ is an irreducible character of $\mathbf{G}(\mathbb{F}_q)$ which is *uniform*, that is, if $\chi$ is in the span of the Deligne–Lusztig virtual characters of $\mathbf{G}(\mathbb{F}_q)$, then $T^*(\chi) = \chi$. Conversely, Asai proves in [2] that if $\mathbf{G}$ is a classical group, and $\chi$ is an irreducible character of $\mathbf{G}(\mathbb{F}_q)$ which is invariant under the twisting operator, then $\chi$ must be uniform. This statement is conjectured to hold for any reductive group $\mathbf{G}$ [15, Conjecture 1.3.3(ii)].

Now let $m > 1$, and let $\chi$ be a $\sigma$-invariant irreducible character of $G = \mathbf{G}(\mathbb{F}_{q^m})$. Since $\chi$ is assumed to be $\sigma$-invariant, then $\chi$ may be extended to an irreducible character of $G\langle\sigma\rangle$. Let $\psi$ be an irreducible character of the group $H = \mathbf{G}(\mathbb{F}_q)$. We say that $\chi$ is a *Shintani lift* of $\psi$ if there exists an irreducible character $\tilde{\chi}$ of $G\langle\sigma\rangle$ extending $\chi$ such that

$$\tilde{\chi}(\sigma[x]_\sigma) = \pm\psi\left(N([x]_\sigma)\right), \quad \text{for all } x \in G, \tag{2.1}$$

where the sign $\pm$ is dependent only on $\chi$ (and is $+$ if $m$ is even), and $N$ is the Shintani norm map $N_{\mathbb{F}_{q^m}/\mathbb{F}_q}$. Shintani [23] first defined this lifting map for $\mathbf{G} = \mathrm{GL}(n, \bar{\mathbb{F}}_q)$, with $\sigma$ the standard Frobenius map, and proved that lifting gives a bijection from the irreducible characters of $\mathrm{GL}(n, \mathbb{F}_q)$ to the irreducible Frobenius-invariant characters of $\mathrm{GL}(n, \mathbb{F}_{q^m})$.

Kawanaka [14,15] has defined a variation of the norm map $N_{\mathbb{F}_{q^m}/\mathbb{F}_q}$ by replacing it with $(T^*_{\mathbb{F}_q})^{-r} \circ N_{\mathbb{F}_{q^m}/\mathbb{F}_q}$, where $r$ is a positive integer depending on $m$ and properties of $\mathbf{G}$. Kawanaka studied lifting [14] by replacing the Shintani norm by this norm in the definition (2.1). This does not change anything in the case that $\psi$ is invariant under the twisting operator, and so by Asai's result, when $\psi$ is a uniform character. It is conjectured [15, Conjecture 1.3.3(i)] that Shintani lifting gives a bijection from irreducible characters of $\mathbf{G}(\mathbb{F}_q)$ which are invariant under the twisting operator $T^*_{\mathbb{F}_q}$ to $\sigma$-invariant irreducible characters of $\mathbf{G}(\mathbb{F}_{q^m})$ which are invariant under the twisting operator $T^*_{\mathbb{F}_{q^m}}$. If we replace the Shintani norm map with Kawanaka's norm map, then it is conjectured [15, Conjecture 1.3.1] that Shintani lifting is a bijection from all irreducible characters of $\mathbf{G}(\mathbb{F}_q)$ to all $\sigma$-invariant irreducible characters of $\mathbf{G}(\mathbb{F}_{q^m})$, under the condition that $m$ is an *admissible* integer for the pair $(\mathbf{G}, \sigma)$ (see [14,15] for the definition of admissible). Kawanaka has proven this statement for finite classical groups [13,14] in the case that the characteristic $p$ of $\mathbb{F}_q$ does not divide $m$.

It will be apparent that all of the results which we obtain in this paper hold whether we use the Shintani norm map, or the norm map of Kawanaka, for the definition of Shintani lifting in (2.1). The only place where this makes a difference is when we apply specific results of Kawanaka for the finite symplectic group, in the last paragraph of Sect. 5, where we are referring to Kawanaka's definition of lifting.

## 3. Lifting real-valued characters

Suppose that $(\pi, V)$ is an irreducible representation of $G = \mathbf{G}(\mathbb{F}_{q^m})$ with character $\chi$, which is $\sigma$-invariant, and let $\psi$ be an irreducible character of $H = \mathbf{G}(\mathbb{F}_q)$ such that $\chi$ is a Shintani lift of $\psi$. Then, we have ${}^{\sigma}\pi \cong \pi$, and so there is a linear transformation $J : V \to V$ such that

$$ {}^{\sigma}\pi(g) \circ J = J \circ \pi(g), \tag{3.1} $$

for every $g \in G$, and since we may extend $\pi$ to an irreducible representation $\tilde{\pi}$ (with character $\tilde{\chi}$, say) of $G\langle\sigma\rangle$, we may choose $J$ such that $J^m = I$. By adjusting $J$ by a sign if necessary, the character definition of Shintani lifting in (2.1) translates to the statement that there exists a transformation $J : V \to V$ satisfying (3.1) such that $J^m = \pm I$, where $J^m = I$ if $m$ is even, and

$$ \mathrm{tr}(J \circ \pi(g)) = \psi(N([g]_{\sigma})), \tag{3.2} $$

for every $g \in G$.

Using the representation version of Shintani lifting in (3.2), and applying Schur orthogonality of matrix coefficients, Bump and Ginzburg [3, Theorem 12] proved the following result. Although they only state the result for $\mathbf{G} = \mathrm{GL}(n, \overline{\mathbb{F}}_q)$ and $\sigma$ the standard Frobenius map, the exact same proof works for the more general situation.

**Theorem 3.1 (Bump and Ginzburg [3]).** *Let $\chi$ be an irreducible character of $G = \mathbf{G}(\mathbb{F}_{q^m})$, and let $\psi$ be an irreducible character of $H = \mathbf{G}(\mathbb{F}_q)$. Then*

$$\frac{1}{|G|}\sum_{g\in G}\chi(g)\overline{\psi(N([g]_\sigma))} = \begin{cases} \psi(1)/\chi(1) & \text{if } \chi \text{ is a Shintani lift of } \psi, \\ 0 & \text{otherwise.}\end{cases}$$

We apply Theorem 3.1 in a crucial way to prove the following, which is the first main result.

**Theorem 3.2.** *Let $\chi$ be an irreducible character of $G = \mathbf{G}(\mathbb{F}_{q^m})$ such that $\chi$ is a Shintani lift of the irreducible character $\psi$ of $H = \mathbf{G}(\mathbb{F}_q)$. Then $\psi$ is real-valued if and only if $\chi$ is real-valued.*

*Proof.* We first show that $\chi$ is the lift of at most one irreducible character of $H$, and the character $\psi$ of $H$ has at most one lift to $G$. These statements follow from well-known results, but we prove them for the sake of completeness. If $\chi$ is a lift of $\psi_1$ and $\psi_2$, then there are two extensions, $\tilde{\chi}$ and $\tilde{\tilde{\chi}}$ of $\chi$ to $G\langle\sigma\rangle$, such that

$$\tilde{\chi}(\sigma[x]_\sigma) = \pm\psi_1\left(N([x]_\sigma)\right) \quad \text{and} \quad \tilde{\tilde{\chi}}(\sigma[x]_\sigma) = \pm\psi_2\left(N([x]_\sigma)\right),$$

for every $x \in G$. By [23, Lemma 1.3], there must exist an $m$-th root of unity, say $\zeta$, such that $\tilde{\chi}(\sigma x) = \zeta\tilde{\tilde{\chi}}(\sigma x)$ for every $x \in G$. This implies that we must have $\psi_1(h) = \pm\zeta\psi_2(h)$ for every $h \in H$ (where the sign $\pm$ is independent of $h$). This is impossible, unless $\psi_1 = \psi_2$, and thus $\chi$ must be a lift of a unique irreducible character of $H$.

Now suppose $\chi_1 = \chi$ and $\chi_2$ are two Shintani lifts of $\psi$, with extensions $\tilde{\chi}_1$ and $\tilde{\chi}_2$ to $G\langle\sigma\rangle$ which satisfy (2.1). If we assume that $\chi_1 = \tilde{\chi}_1|_G \neq \tilde{\chi}_2|_G = \chi_2$, then by [23, Lemma 1.2], we have

$$\sum_{g\in G}\tilde{\chi}_1(\sigma g)\overline{\tilde{\chi}_2(\sigma g)} = 0. \tag{3.3}$$

However, from the assumption that $\chi_1$ and $\chi_2$ are both Shintani lifts of $\psi$, we have $\tilde{\chi}_1(\sigma g) = \pm\tilde{\chi}_2(\sigma g)$ for all $g \in G$, where the sign $\pm$ is independent of $g$. This would imply

$$\sum_{g\in G}\tilde{\chi}_1(\sigma g)\overline{\tilde{\chi}_2(\sigma g)} = \pm\sum_{g\in G}|\tilde{\chi}_1(\sigma g)|^2 \neq 0,$$

which contradicts (3.3). Therefore, $\chi_1 = \chi_2$, and $\psi$ can have at most one Shintani lift to $G$.

Now assume that $\chi$ is real-valued, and is a Shintani lift of $\psi$. By Theorem 3.1, we have

$$\frac{1}{|G|}\sum_{g\in G}\chi(g)\overline{\psi(N([g]_\sigma))} = \frac{\psi(1)}{\chi(1)}. \tag{3.4}$$

If we conjugate both sides of (3.4), then since $\chi$ is assumed to be real-valued, we obtain

$$\frac{1}{|G|}\sum_{g\in G}\chi(g)\overline{\overline{\psi}(N([g]_\sigma))} = \frac{\psi(1)}{\chi(1)} = \frac{\overline{\psi}(1)}{\chi(1)}.$$

By Theorem 3.1 again, we must then have that $\chi$ is a Shintani lift of the irreducible character $\overline{\psi}$. However, as we have shown above, $\chi$ is the lift of at most one irreducible character of $H$, and so we must have $\psi = \overline{\psi}$, and so $\psi$ is real-valued. Conversely, if we assume $\psi$ is real-valued, then a similar argument shows that $\chi$ and $\overline{\chi}$ are both Shintani lifts of $\psi$, which implies that $\chi = \overline{\chi}$ is real-valued. □

Gyoja and Lusztig (see [12, Section 9]) both pointed out that the group $\mathrm{Sp}(4, \mathbb{F}_q)$ has irreducible unipotent characters which do not have Shintani lifts to $\mathrm{Sp}(4, \mathbb{F}_{q^2})$. We may apply Theorem 3.2, along with reality results for $\mathrm{Sp}(2n, \mathbb{F}_q)$, to obtain a general class of characters of the finite symplectic groups which do not have even degree Shintani lifts.

**Corollary 3.1.** *Let $m$ be even, $q \equiv 3(mod\ 4)$, and $\psi$ an irreducible character of* $\mathrm{Sp}(2n, \mathbb{F}_q)$ *which is not real-valued. Then $\psi$ has no Shintani lift to* $\mathrm{Sp}(2n, \mathbb{F}_{q^m})$.

*Proof.* It follows from [9, Lemma 5.3] that when $q \equiv 3(\mathrm{mod}\ 4)$, the group $\mathrm{Sp}(2n, \mathbb{F}_q)$ has irreducible characters which are not real-valued. If $\psi$ is such a character, then by Theorem 3.2, any lift of $\psi$ to $\mathrm{Sp}(2n, \mathbb{F}_{q^m})$ must also take non-real values. However, since $m$ is even, $q^m \equiv 1(\mathrm{mod}\ 4)$, and it follows from results of Wonenburger [27, Theorem 2] that in this case, all irreducible characters of $\mathrm{Sp}(2n, \mathbb{F}_{q^m})$ must be real-valued, and so $\psi$ cannot have a Shintani lift. □

Now consider the case $\mathbf{G}(\bar{\mathbb{F}}_q) = \mathrm{GL}(n, \bar{\mathbb{F}}_q)$, with the standard Frobenius map $\sigma$, defined by ${}^\sigma(x_{ij}) = (x_{ij}^q)$, and with the standard Frobenius map composed with the inverse-transpose automorphism (or the *twisted* Frobenius), which we denote by $\tilde{\sigma}$, so ${}^{\tilde{\sigma}}(x_{ij}) = (x_{ji}^q)^{-1}$. Then $\mathbf{G}(\bar{\mathbb{F}}_q)^\sigma = \mathrm{GL}(n, \mathbb{F}_q)$ and $\mathbf{G}(\bar{\mathbb{F}}_q)^{\tilde{\sigma}} = \mathrm{U}(n, \mathbb{F}_q)$, the finite unitary group, are both contained in $\mathbf{G}(\bar{\mathbb{F}}_q)^{\sigma^2} = \mathbf{G}(\bar{\mathbb{F}}_q)^{\tilde{\sigma}^2} = \mathrm{GL}(n, \mathbb{F}_{q^2})$. By the original results of Shintani [23], lifting is a bijection from the irreducible characters of $\mathrm{GL}(n, \mathbb{F}_q)$ to the $\sigma$-invariant irreducible characters of $\mathrm{GL}(n, \mathbb{F}_{q^2})$, a map which we will denote by **sh**. By results of Kawanaka [13], Shintani lifting gives a bijective map (when $q$ is odd) from irreducible characters of $\mathrm{U}(n, \mathbb{F}_q)$ to the $\tilde{\sigma}$-invariant irreducible characters of $\mathrm{GL}(n, \mathbb{F}_{q^2})$, which we denote by $\widetilde{\mathbf{sh}}$. By Theorem 3.2, both of these maps restrict to bijections of the subsets of real-valued characters. Also, irreducible characters of $\mathrm{GL}(n, \mathbb{F}_{q^2})$ which are invariant under $\sigma$ and $\tilde{\sigma}$ are invariant under the inverse-transpose map, which are thus real-valued, since every element of $\mathrm{GL}(n, \mathbb{F}_{q^2})$ is conjugate to its transpose, and $\chi(g^{-1}) = \overline{\chi(g)}$. From these observations, we obtain the following.

**Corollary 3.2.** *Let $q$ be the power of an odd prime. Then the map $\widetilde{\mathbf{sh}}^{-1} \circ \mathbf{sh}$ is a bijection from the irreducible real-valued characters of* $\mathrm{GL}(n, \mathbb{F}_q)$ *to the irreducible real-valued characters of* $\mathrm{U}(n, \mathbb{F}_q)$.

We remark that Gow [10] showed that the number of irreducible real-valued characters of $\mathrm{GL}(n, \mathbb{F}_q)$ is equal to the number of irreducible real-valued characters of $\mathrm{U}(n, \mathbb{F}_q)$, by giving a bijection between the real conjugacy classes of these groups.

## 4. Quadratic Shintani lifting

For a finite group $G$, with a complex representation $(\pi, V)$, we say that $\pi$ is a *real representation* of $G$ if there is a basis for $V$ such that for every $g \in G$, the matrix for $\pi(g)$ with respect to this basis is a matrix with all real entries. For any irreducible complex representation $\pi$ of $G$ with character $\chi$, let $\varepsilon(\chi)$ (or $\varepsilon(\pi)$) denote the Frobenius–Schur indicator of $\chi$, which takes the value 1 if $(\pi, V)$ is a real representation, $-1$ if $\chi$ is real-valued but $(\pi, V)$ is not a real representation, and 0 if $\chi$ is not real-valued. If $\varepsilon(\chi) = 1$, $\chi$ is called *orthogonal*, and if $\varepsilon(\chi) = -1$, $\chi$ is called *symplectic*. It is a classical result of Frobenius and Schur that $\varepsilon(\chi) = (1/|G|) \sum_{g \in G} \chi(g^2)$.

Now let $\tau$ be an automorphism of $G$ such that $\tau^2$ is the identity map. If $\chi$ is an irreducible character of $G$, define the *twisted Frobenius–Schur indicator* of $\chi$ with respect to $\tau$, denoted by $\varepsilon_\tau(\chi)$, to be the sum $\varepsilon_\tau(\chi) = (1/|G|) \sum_{g \in G} \chi({}^\tau g g)$. When $\tau$ is the identity automorphism, we just have $\varepsilon_\tau(\chi) = \varepsilon(\chi)$. It is a result of Kawanaka and Matsuyama [17] that the invariant $\varepsilon_\tau(\chi)$ has the following characterization, making it a natural generalization of the original indicator of Frobenius and Schur. If $\chi$ is the character of the irreducible complex representation $(\pi, V)$ of $G$, then $\varepsilon_\tau(\chi) = 1$ if there exists a basis for $V$ such that the matrix representation $R$ for $\pi$ with respect to this basis satisfies $R({}^\tau g) = \overline{R(g)}$ for every $g \in G$, $\varepsilon_\tau(\chi) = -1$ if ${}^\tau \chi = \overline{\chi}$ but there does not exist such a basis for $V$, and $\varepsilon_\tau(\chi) = 0$ if ${}^\tau \chi \neq \overline{\chi}$.

Consider the split extension $G\langle\tau\rangle$ of $G$ by an order 2 automorphism $\tau$, and an irreducible character $\chi$ of $G$. It follows from Frobenius reciprocity that the induced character $\chi^{G\langle\tau\rangle}$ is reducible if and only if $\chi$ is $\tau$-invariant, that is, ${}^\tau \chi = \chi$. In the case that $\chi$ is $\tau$-invariant, we have $\chi^{G\langle\tau\rangle} = \tilde{\chi} + \tilde{\tilde{\chi}}$, where $\tilde{\chi}$ and $\tilde{\tilde{\chi}}$ are both irreducible characters of $G\langle\tau\rangle$ which extend $\chi$. The following result comes directly from these facts and the formulas for the standard and twisted Frobenius–Schur indicators given in the previous paragraphs.

**Lemma 4.1.** *Let $\chi$ be an irreducible character of $G$, and let $\tau$ be an order 2 automorphism of $G$. If ${}^\tau \chi = \chi$, then for any irreducible extension $\tilde{\chi}$ of $\chi$ to $G\langle\tau\rangle$, we have*

$$\varepsilon(\tilde{\chi}) = \frac{1}{2}(\varepsilon(\chi) + \varepsilon_\tau(\chi)).$$

We now consider the situation when $G = \mathbf{G}(\mathbb{F}_{q^2})$, and $H = \mathbf{G}(\mathbb{F}_q) = G^\sigma$, where $\mathbf{G}$ is a connected reductive linear algebraic group defined over $\mathbb{F}_q$ with Frobenius map $\sigma$. Then $\sigma$ is an order 2 automorphism of $G$, and we may consider the indicator $\varepsilon_\sigma(\chi)$ of an irreducible character $\chi$ of $G$. We will need the following result of Kawanaka [16], which relates the indicators $\varepsilon_\sigma(\chi)$ to the decomposition of the permutation character of $G$ on $H$.

**Theorem 4.1 (Kawanaka [16]).** *Let $\chi$ be an irreducible character of $G = \mathbf{G}(\mathbb{F}_{q^2})$ which is invariant under the twisting operator $T^*_{\mathbb{F}_{q^2}}$, and let $H = \mathbf{G}(\mathbb{F}_q)$. Then the multiplicity of $\chi$ in the permutation character of $\mathbf{G}(\mathbb{F}_{q^2})$ on $\mathbf{G}(\mathbb{F}_q)$ is equal to $\varepsilon_\sigma(\chi)$, that is, $\langle \chi, \mathbf{1}_H^G \rangle = \varepsilon_\sigma(\chi)$.*

Theorem 4.1 was first proved in the case that $G = \mathrm{GL}(n, \mathbb{F}_{q^2})$, and $\sigma$ is either the standard Frobenius or the twisted Frobenius, by Gow [10]. Lusztig [18] improved on the results of Kawanaka [16] by finding multiplicities of any character in the permutation character, while Prasad [21] proved results similar to Theorem 4.1, under a weaker assumption on the underlying algebraic group. Shoji and Sorlin [25, Section 1] defined an indicator for an automorphism of degree $m \geq 2$ (similar to those considered by Bump and Ginzburg [3]), and related them to the permutation character of $\mathbf{G}(\mathbb{F}_{q^m})$ on $\mathbf{G}(\mathbb{F}_q)$ through Shintani descent. In the quadratic case, because of the close relationship between the twisted and standard Frobenius–Schur indicators, we are able to obtain the following result.

**Theorem 4.2.** *Let $\chi$ be an irreducible character of $\mathbf{G}(\mathbb{F}_{q^2})$, invariant under the twisting operator, which is the Shintani lift of an irreducible real-valued character of $\mathbf{G}(\mathbb{F}_q)$. Then $\varepsilon(\chi) = 1$.*

*Proof.* Let $G = \mathbf{G}(\mathbb{F}_{q^2})$, $H = \mathbf{G}(\mathbb{F}_q)$, and let $\psi$ be the irreducible real-valued character of $H$ such that $\chi$ is the Shintani lift of $\psi$. By Theorem 3.2, $\chi$ must be real-valued as well. Let $\tilde{\chi}$ be the extension of $\chi$ to $G\langle\sigma\rangle$ such that $\tilde{\chi}(\sigma g) = \psi(N([g]_\sigma))$ for all $g \in G$. Since $\psi$ is real-valued, then $\tilde{\chi}$ is real-valued on the coset $G\sigma$, and since $\tilde{\chi}$ extends $\chi$, which is real-valued as well, then $\tilde{\chi}$ is real-valued, and so $\varepsilon(\tilde{\chi}) = \pm 1$.

Since $\chi$ is the Shintani lift of $\psi$, then in particular $\chi$ is $\sigma$-invariant, and since $\chi$ is real-valued, then ${}^\sigma\chi = \overline{\chi}$, and so $\varepsilon_\sigma(\chi) = \pm 1$. By Theorem 4.1, since $\chi$ is invariant under the twisting operator, then $\varepsilon_\sigma(\chi)$ is given by the multiplicity of $\chi$ in $\mathbf{1}_H^G$, and thus must be non-negative. We must then have $\varepsilon_\sigma(\chi) = 1$. From Lemma 4.1, we have

$$\varepsilon(\tilde{\chi}) = \frac{1}{2}(\varepsilon(\chi) + 1).$$

We have shown that $\varepsilon(\tilde{\chi}) = \pm 1$, and since $\varepsilon(\chi) = \pm 1$ as well, the only possibility is $\varepsilon(\tilde{\chi}) = 1$ and $\varepsilon(\chi) = 1$, as desired. □

Theorem 4.2 may be paraphrased as saying that real-valued Frobenius-invariant characters of $\mathbf{G}(\mathbb{F}_{q^2})$ tend to be orthogonal. We do not know of an example of an irreducible real-valued character of a group $\mathbf{G}(\mathbb{F}_{q^2})$ which is in the image of Shintani lifting, not invariant under the twisting operator, and yet is symplectic rather than orthogonal. If there is such an example, then it will most likely be a character of an exceptional group (see Corollary 5.1 and the remarks preceding it).

## 5. Central elements, regular and semisimple characters

For any finite group $K$, we let $Z(K)$ denote the center of $K$. If $(\pi, V)$ is an irreducible representation of $K$, then it follows from Schur's Lemma that $\pi(z)$ acts as a scalar on $V$ for any $z \in Z(K)$. We let $\omega_\pi(z)$ denote this scalar, or $\omega_\chi(z)$ where $\chi$ is the character of $\pi$, so that $\omega_\chi : K \to \mathbb{C}^\times$ is a multiplicative homomorphism called the central character for $\pi$ (or $\chi$). We first observe that the central character behaves nicely under Shintani lifting.

**Lemma 5.1.** *Let $\chi$ be an irreducible character of $\mathbf{G}(\mathbb{F}_{q^m})$ which is the Shintani lift of the irreducible character $\psi$ of $\mathbf{G}(\mathbb{F}_q)$, and let $z \in Z(\mathbf{G}(\mathbb{F}_q))$. Then $\omega_\chi(z) = \omega_\psi(z)^m$.*

*Proof.* Since $\chi$ is a Shintani lift of $\psi$, then there is an extension $\tilde{\chi}$ of $\chi$ to $G\langle\sigma\rangle$ such that $\tilde{\chi}(\sigma x) = \pm\psi(N([x]_\sigma))$ for every $x \in G$. In particular, since $N([1]_\sigma) = [1]$, then $\tilde{\chi}(\sigma) = \pm\psi(1)$. By [4, Proposition 3.6.8], $Z(\mathbf{G}(\bar{\mathbb{F}}_q))^\sigma = Z(\mathbf{G}(\mathbb{F}_q))$, and so $z \in Z(\mathbf{G}(\bar{\mathbb{F}}_q))$. If we choose $\alpha_z \in \mathbf{G}(\bar{\mathbb{F}}_q)$ such that ${}^\sigma\alpha_z^{-1}\alpha_z = z$, then

$$\alpha_z\,({}^{\sigma^m}\alpha_z^{-1}) = \alpha_z\left(\prod_{i=1}^{m}({}^{\sigma^{m-i}}z)\right)\alpha_z^{-1} = \alpha_z z^m \alpha_z^{-1} = z^m.$$

That is, $N([z]_\sigma) = [z^m]$, and so $\tilde{\chi}(\sigma z) = \pm\psi(z^m)$. Now, we have

$$\pm\psi(z^m) = \pm\omega_\psi(z^m)\psi(1) = \tilde{\chi}(\sigma z) = \omega_\chi(z)\tilde{\chi}(\sigma),$$

and since $\tilde{\chi}(\sigma) = \pm\psi(1)$, we have $\omega_\chi(z) = \omega_\psi(z)^m$. □

Note that the proof of Lemma 5.1 is the same if we consider lifting defined using Kawanaka's norm, since central elements are invariant under the map $T_{\mathbb{F}_q}$.

We now consider the case when $\mathbf{G}$ is a connected reductive group with connected center which is defined over $\mathbb{F}_q$. An irreducible character $\chi$ of $\mathbf{G}(\mathbb{F}_q)$ is called a *regular* character if it appears in the decomposition into irreducibles of the Gelfand–Graev character of $\mathbf{G}(\mathbb{F}_q)$, which is obtained by inducing a non-degenerate linear character from a maximal unipotent subgroup. A *semisimple* character of $\mathbf{G}(\mathbb{F}_q)$ is an irreducible character which takes a nonzero average value on regular unipotent elements. Equivalently, a semisimple character is an irreducible which appears in the decomposition of the character obtained by applying the *duality functor* (or *Alvis–Curtis dual*) to the Gelfand–Graev character of $\mathbf{G}(\mathbb{F}_q)$. That is, semisimple characters are those which are obtained, up to a sign, when applying the duality functor to regular characters. When $p = \mathrm{char}(\mathbb{F}_q)$ is a good prime for $\mathbf{G}$, then the semisimple characters of $\mathbf{G}(\mathbb{F}_q)$ are exactly the irreducible characters with degree not divisible by $p$. For a discussion of regular characters, semisimple characters, and the duality functor, see [8, Chaps. 8 and 14] or [4, Chap. 8].

If a regular or semisimple character is real-valued, then there is always a central element which controls the Frobenius–Schur indicator of the character. The first version of the following result was obtained by Prasad [21, Theorem 3], and was improved to the following form by the author [26, Theorem 4.2].

**Theorem 5.1 (Prasad [21], Vinroot [26]).** *Let* $\mathbf{G}$ *be a connected reductive group with connected center over* $\bar{\mathbb{F}}_q$*, defined over* $\mathbb{F}_q$*. Then there exists an element* $z \in Z(\mathbf{G}(\mathbb{F}_q))$ *such that, for any* $m \geq 1$*, and any real-valued regular or semisimple character* $\chi$ *of* $\mathbf{G}(\mathbb{F}_{q^m})$*, we have* $\varepsilon(\chi) = \omega_\chi(z)$*.*

We now apply the results above, along with a result of Gyoja [12] to see that the Frobenius–Schur indicators of regular and semisimple character behave nicely under Shintani lifting.

**Theorem 5.2.** *Let* $\mathbf{G}$ *be a connected reductive group with connected center which is defined over* $\mathbb{F}_q$*. Let* $\chi$ *be an irreducible character of* $\mathbf{G}(\mathbb{F}_{q^m})$ *which is a Shintani lift of a regular or semisimple real-valued character* $\psi$ *of* $\mathbf{G}(\mathbb{F}_q)$*. Then*

$$\varepsilon(\chi) = \begin{cases} \varepsilon(\psi) & \textit{if } m \textit{ is odd,} \\ 1 & \textit{if } m \textit{ is even.} \end{cases}$$

*Proof.* Since $\psi$ is a real-valued character, then by Theorem 3.2, $\chi$ is also real-valued. Gyoja proved that if $\psi$ is regular, and has a Shintani lift $\chi$, then $\chi$ must also be regular, and if $\psi$ is semisimple, then $\chi$ must also be semisimple [12, Lemmas 6.3 and 6.4]. By Theorem 5.1, there exists an element $z \in Z(\mathbf{G}(\mathbb{F}_q))$ such that $\varepsilon(\chi) = \omega_\chi(z)$ and $\varepsilon(\psi) = \omega_\psi(z)$. Finally, by Lemma 5.1 we have $\varepsilon(\chi) = \omega_\chi(z) = \omega_\psi(z)^m = \varepsilon(\psi)^m$, which is equal to 1 when $m$ is even, and $\varepsilon(\psi)$ when $m$ is odd. □

We suspect that Theorem 5.2 holds for any real-valued irreducible $\chi$ which is a Shintani lift, or at least for those characters which are invariant under the twisting operator on characters of $\mathbf{G}(\mathbb{F}_{q^m})$. This is the content of Theorem 4.2 in the case $m = 2$, without the assumption that the center is connected.

Suppose that Theorem 5.2 holds for some irreducible character $\chi$, which is the lift of $\psi$, and let $\chi^*$ denote the irreducible character obtained by applying the duality functor to $\chi$ (where the correct sign is chosen to obtain a true character). McGovern [19, Theorem 4.3] has shown that $\chi^*$ is then the Shintani lift of $\psi^*$. By a result of the author [26, Theorem 3.2], we also have $\varepsilon(\chi) = \varepsilon(\chi^*)$ and $\varepsilon(\psi) = \varepsilon(\psi^*)$, which implies Theorem 5.2 would also hold for the character $\chi^*$. That is, if the conclusion of Theorem 5.2 holds for an irreducible character $\chi$, then it holds also for the character $\chi^*$.

There are several groups for which it is known that all real-valued irreducible characters are characters of real representations, such as $\mathrm{GL}(n, \mathbb{F}_q)$ (by [21, Theorem 4], for example) and $\mathrm{SO}^{\pm}(n, \mathbb{F}_q)$ for $q$ odd (by [11, Theorem 2]). In these cases, there is nothing to check for the conclusion of Theorem 5.2 to hold for all irreducible characters in the image of Shintani lifting. On the other hand, for the finite symplectic group $\mathrm{Sp}(2n, \mathbb{F}_q)$, when $q$ is the power of an odd prime, Gow [11, Theorem 1] proved that every irreducible real-valued character $\chi$ satisfies $\varepsilon(\chi) = \omega_\chi(-I)$. In this case, we may apply Lemma 5.1 and Theorem 3.2 to obtain the conclusion in Theorem 5.2 for all irreducible characters in the image of Shintani lifting, as in the following.

**Corollary 5.1.** *Let $q$ be the power of an odd prime. Let $\chi$ be an irreducible character of* $\mathrm{Sp}(2n, \mathbb{F}_{q^m})$ *which is the Shintani lift of the real-valued irreducible character $\psi$ of* $\mathrm{Sp}(2n, \mathbb{F}_q)$*. Then*

$$\varepsilon(\chi) = \begin{cases} \varepsilon(\psi) & \text{if } m \text{ is odd,} \\ 1 & \text{if } m \text{ is even.} \end{cases}$$

When $m$ is odd and relatively prime to $q$, Kawanaka [14] proved that lifting gives a bijection from irreducible characters of $\mathrm{Sp}(2n, \mathbb{F}_q)$ to the Frobenius-invariant irreducible characters of $\mathrm{Sp}(2n, \mathbb{F}_{q^m})$. It follows from Corollary 5.1 and Theorem 3.2 that in this case, if $\psi$ is any irreducible character of $\mathrm{Sp}(2n, \mathbb{F}_q)$, and $\chi$ is its Shintani lift to $\mathrm{Sp}(2n, \mathbb{F}_{q^m})$, then $\varepsilon(\chi) = \varepsilon(\psi)$.

## 6. Deligne–Lusztig characters

Let $\mathbf{G}$ be a connected reductive group over $\bar{\mathbb{F}}_q$, with connected center, defined over $\mathbb{F}_q$ with Frobenius map $\sigma$. If $\mathbf{T}$ is a $\sigma$-stable maximal torus of $\mathbf{G}$, and $\theta$ a multiplicative character of $\mathbf{T}(\mathbb{F}_q)$, we let $R_{\mathbf{T},\theta}$ denote the Deligne–Lusztig virtual character (see [5]) of $\mathbf{G}(\mathbb{F}_q)$ corresponding to the pair $(\mathbf{T}, \theta)$. Recall that $\pm R_{\mathbf{T},\theta}$ is irreducible if and only if $(\mathbf{T}, \theta)$ is in general position, that is, there is no non-trivial element of the Weyl group $W(\mathbf{T})^\sigma$ which maps $(\mathbf{T}, \theta)$ to itself. If $(\mathbf{T}', \theta')$ is another pair, and $\pm R_{\mathbf{T},\theta}$ is irreducible, then $R_{\mathbf{T},\theta} = R_{\mathbf{T}',\theta'}$ if and only if there is a unique element $w \in W(\mathbf{T})^\sigma$ such that ${}^w\mathbf{T} = \mathbf{T}'$ and ${}^w\theta = \theta'$ (see [4, Theorem 7.3.4]).

Given a $\sigma$-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$, we have $\mathbf{T}$ is also a $\sigma^m$-stable maximal torus for any $m > 1$, and so we may consider both of the finite groups $\mathbf{T}(\mathbb{F}_q)$ and $\mathbf{T}(\mathbb{F}_{q^m})$. Define a homomorphism, which we denote by $\mathrm{Nm}_{\mathbb{F}_{q^m}/\mathbb{F}_q}$ (or simply Nm), by

$$\mathrm{Nm}_{\mathbb{F}_{q^m}/\mathbb{F}_q} : \mathbf{T}(\mathbb{F}_{q^m}) \to \mathbf{T}(\mathbb{F}_q), \quad \mathrm{Nm}(x) = \prod_{i=0}^{m-1} ({}^{\sigma^{m-i}} x).$$

Since $\mathbf{T}$ is itself a connected reductive group, which happens to be abelian, the only difference between the map Nm and the map $N_{\mathbb{F}_{q^m}/\mathbb{F}_q}$ in Proposition 2.1 is that we have defined Nm element-wise, rather than on $\sigma$-conjugacy classes. In particular, by Proposition 2.1, Nm is a surjective homomorphism with fibers being $\sigma$-conjugacy classes of $\mathbf{T}(\mathbb{F}_{q^m})$.

If $\theta$ is a multiplicative homomorphism of $\mathbf{T}(\mathbb{F}_q)$, where $\mathbf{T}$ is a $\sigma$-stable maximal torus of $\mathbf{G}$, then $\theta \circ \mathrm{Nm}$ is a multiplicative homomorphism of $\mathbf{T}(\mathbb{F}_{q^m})$. Gyoja studied the map $(\mathbf{T}, \theta) \mapsto (\mathbf{T}, \theta \circ \mathrm{Nm})$, and showed that it induces a map from irreducible Deligne–Lusztig characters $\pm R_{\mathbf{T},\theta}$ of $\mathbf{G}(\mathbb{F}_q)$ to $\sigma$-invariant irreducible Deligne–Lusztig characters of $\mathbf{G}(\mathbb{F}_{q^m})$ [12, Lemmas 6.6 and 6.9]. Under certain conditions on $p = \mathrm{char}(\mathbb{F}_q)$, $q$, and $m$, Gyoja [12, Theorems 7.2 and 8.4] proved that this map coincides with Shintani lifting from $\mathbf{G}(\mathbb{F}_q)$ to $\mathbf{G}(\mathbb{F}_{q^m})$. Digne [6, Corollaire 3.6] proved a generalization of Gyoja's result for Shintani descent of virtual Deligne–Lusztig characters, under the conditions that $p$ is a good prime for $\mathbf{G}$ and $(m, p) = 1$. Since irreducible Deligne–Lusztig characters are both regular and semisimple (see [4, Section 8.4], for example), we could apply Theorem 5.2 to make conclusions about real-valued irreducible Deligne–Lusztig characters under these conditions. However, we are able to prove these results independent of the results on Shintani lifting of irreducible Deligne–Lusztig characters, and also under no conditions on $p$, $q$, or $m$. That is, we prove directly that real-valued irreducible Deligne–Lusztig behave nicely under the bijection induced by the map $(\mathbf{T}, \theta) \mapsto (\mathbf{T}, \theta \circ \mathrm{Nm})$. We conclude with the following result, which improves [26, Theorem 5.1].

**Theorem 6.1.** *Let $\psi = \pm R_{\mathbf{T},\theta}$ be an irreducible Deligne–Lusztig character of $\mathbf{G}(\mathbb{F}_q)$, and let $\chi = \pm R_{\mathbf{T},\Theta}$ be the irreducible Deligne–Lusztig character of $\mathbf{G}(\mathbb{F}_{q^m})$, where $\Theta : \mathbf{T}(\mathbb{F}_{q^m}) \to \mathbb{C}^\times$ is defined as $\Theta = \theta \circ \mathrm{Nm}$. Then $\psi$ is real-valued if and only if $\chi$ is real-valued. When they are real-valued, we have*

$$\varepsilon(\chi) = \begin{cases} \varepsilon(\psi) & \text{if } m \text{ is odd,} \\ 1 & \text{if } m \text{ is even.} \end{cases}$$

*Proof.* If $\psi$ is real-valued, then $R_{\mathbf{T},\theta} = \overline{R_{\mathbf{T},\theta}} = R_{\mathbf{T},\overline{\theta}}$ by [8, Proposition 11.4], and so there is a unique $s \in W(\mathbf{T})^\sigma$ such that ${}^s\theta = \overline{\theta}$. This implies $s^2 = 1$, since ${}^{s^2}\theta = \theta$ and $\psi$ is assumed to be irreducible. For any $y \in \mathbf{T}(\mathbb{F}_{q^m})$, we have $\mathrm{Nm}(sys) = s\mathrm{Nm}(y)s$, since ${}^\sigma s = s$. Now,

$${}^s\Theta(y) = \theta(s\mathrm{Nm}(y)s) = {}^s\theta(\mathrm{Nm}(y)) = \overline{\Theta}(y).$$

So, ${}^s\Theta = \overline{\Theta}$, where $s \in W(\mathbf{T})^\sigma \subseteq W(\mathbf{T})^{\sigma^m}$. Thus $\chi = \pm R_{\mathbf{T},\Theta}$ is real-valued.

If $\chi$ is real-valued, then there is a unique $w \in W(\mathbf{T})^{\sigma^m}$ such that ${}^w\Theta = \overline{\Theta}$, and again we have $w^2 = 1$. We claim that ${}^\sigma w = w$, and to show this it is enough to prove that ${}^{\sigma w}\Theta = \overline{\Theta}$. For any $y \in \mathbf{T}(\mathbb{F}_{q^m})$, we have $\mathrm{Nm}({}^\sigma w y {}^\sigma w) = \mathrm{Nm}(w({}^{\sigma^{m-1}} y)w)$. Now,

$${}^{\sigma w}\Theta(y) = \theta(\mathrm{Nm}(w({}^{\sigma^{m-1}} y)w)) = {}^w\Theta({}^{\sigma^{m-1}} y) = \overline{\Theta}({}^{\sigma^{m-1}} y) = \overline{\Theta}(y),$$

since $\Theta$ is $\sigma$-invariant. We now have $w \in W(\mathbf{T})^\sigma$, as claimed. For any $x \in \mathbf{T}(\mathbb{F}_q)$, we may choose a $y \in \mathbf{T}(\mathbb{F}_{q^m})$ such that $\mathrm{Nm}(y) = x$, since Nm is surjective, and so

$${}^w\theta(x) = \theta(w\mathrm{Nm}(y)w) = {}^w\Theta(y) = \overline{\Theta}(y) = \overline{\theta}(x).$$

Since ${}^w\theta = \overline{\theta}$, we have $\psi$ is real-valued.

If $\chi$ and $\psi$ are real valued, then by Theorem 5.1, there is an element $z \in Z(\mathbf{G}(\mathbb{F}_q))$ such that $\varepsilon(\chi) = \omega_\chi(z)$ and $\varepsilon(\psi) = \omega_\psi(z)$. It follows from [4, Proposition 7.5.3 and Theorem 7.5.1] that $\omega_\chi(z) = \Theta(z)$ and $\omega_\psi(z) = \theta(z)$ (see also [26, Lemma 5.1]). Since $\mathrm{Nm}(z) = z^m$, we have

$$\varepsilon(\chi) = \Theta(z) = \theta(\mathrm{Nm}(z)) = \theta(z)^m = \varepsilon(\psi)^m,$$

which is $\varepsilon(\psi)$ if $m$ is odd, and 1 if $m$ is even. □

*Acknowledgements.* The author thanks Jeff Adler for several helpful discussions, and the referee for carefully reading this paper. This research was supported by NSF grant DMS-0854849.

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