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Turull has described the fields of values for characters of $\mathrm{SL}_n(q)$ in terms of the parametrization of the characters of $\mathrm{GL}_n(q)$. In this article, we extend these results to the case of $\mathrm{SU}_n(q)$.

1. Introduction

It is a problem of general interest to understand the fields of values of the complex characters of finite groups, as these fields often reflect important or subtle properties of the group itself. Turull [2001, Section 4] computed the fields of character values of the finite special linear groups $\mathrm{SL}_n(q)$ by using properties of Zelevinsky's degenerate Gelfand–Graev characters of $\mathrm{GL}_n(q)$. In this paper, we extend these methods to compute the fields of character values for the finite special unitary groups $\mathrm{SU}_n(q)$. In particular, we use properties of Kawanaka's generalized Gelfand–Graev characters of $\mathrm{SU}_n(q)$ and the full unitary group $\mathrm{GU}_n(q)$ to get this information. Further, we frame these methods so that we obtain many results for both $\mathrm{SL}_n(q)$ and $\mathrm{SU}_n(q)$ simultaneously.

Turull also computes the Schur indices of the characters of $\mathrm{SL}_n(q)$. This appears to be a much more difficult problem for $\mathrm{SU}_n(q)$. For example, it is helpful in the $\mathrm{SL}_n(q)$ case that the Schur index for every character of $\mathrm{GL}_n(q)$ is 1. However, the Schur indices of the characters of $\mathrm{GU}_n(q)$ are not all explicitly known, but are known to take values other than 1.

This paper is organized as follows. In Section 2, we establish the necessary results from character theory that are needed for the main arguments. In Sections 2A and 2B, we give some tools from Deligne–Lusztig theory and the parametrization of the characters of $\mathrm{GL}_n^\epsilon(q)$, respectively, and we use these to describe the characters of $\mathrm{SL}_n^\epsilon(q)$ in Section 2C. We introduce generalized Gelfand–Graev characters in Section 2D. In Section 3, we obtain some preliminary results on fields of character values which follow quickly from the material in Section 2. To deal with the harder cases, we need some explicit information on unipotent elements in unitary

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groups obtained in Section 4, and we apply this information to generalized Gelfand–Graev characters in Section 5. Finally, in Section 6 we prove our main results in Theorem 6.1 and Corollary 6.3, which give explicitly the field of values of any character of $\mathrm{SU}_n(q)$ and a description of the real-valued characters of $\mathrm{SU}_n(q)$. In particular, these results can be phrased in the same way as the corresponding results for $\mathrm{SL}_n(q)$ found in [Turull 2001, Section 4], so that we may state both our results and Turull's simultaneously.

Notation. We will often use the notation found in [Turull 2001], for clarity of analogous statements. For example, the natural action of a Galois automorphism $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on a character χ of a group will be denoted $\sigma\chi$. Here for a group element g, the value of $\sigma\chi$ is given by $\sigma\chi(g) = \sigma(\chi(g))$. We write $\mathbb{Q}(\chi)$ for the field obtained from $\mathbb{Q}$ by adjoining all values of the character χ.

For an integer n, we will write $n = n_2 n_{2'}$ where n_2 is a 2-power and $n_{2'}$ is odd. Further, for an element x of a finite group Y, we write $x = x_2 x_{2'}$ where x_2 has 2-power order and $x_{2'}$ has odd order. We denote by $|x|$ the order of the element x (we also use this notation for cardinality and size of partitions, which will be clear from context). We write $\mathrm{Irr}(Y)$ for the set of all irreducible complex characters of the finite group Y. Given two elements g, x in Y, we write $g^x = x^{-1}gx$, and for $\chi \in \mathrm{Irr}(Y)$, we define χ^x by $\chi^x(g) = \chi(xgx^{-1})$.

For a subgroup $X \le Y$, we write $\mathrm{Ind}_X^Y(\varphi)$ for the character of Y induced from a character φ of X, and we write $\mathrm{Res}_X^Y(\chi)$ for the character of X restricted from a character χ of Y. We will further use $\mathrm{Irr}(Y|\varphi)$ and $\mathrm{Irr}(X|\chi)$ to denote the set of irreducible constituents of $\mathrm{Ind}_X^Y(\varphi)$ and $\mathrm{Res}_X^Y(\chi)$, respectively.

Throughout the article, let q be a power of a prime p and let $G = \mathrm{SL}_n^\epsilon(q)$ and $\widetilde{G} = \mathrm{GL}_n^\epsilon(q)$, where $\epsilon \in \{\pm 1\}$. Here when $\epsilon = 1$, we mean $\widetilde{G} = \mathrm{GL}_n(q)$ and $G = \mathrm{SL}_n(q)$, and when $\epsilon = -1$, we mean $\widetilde{G} = \mathrm{GU}_n(q)$ and $G = \mathrm{SU}_n(q)$. We also write $\boldsymbol{G} = \mathrm{SL}_n(\overline{\mathbb{F}}_q)$ and $\widetilde{\boldsymbol{G}} = \mathrm{GL}_n(\overline{\mathbb{F}}_q)$ for the corresponding algebraic groups, so that $\widetilde{G} = \widetilde{\boldsymbol{G}}^{F_\epsilon}$ and $G = \boldsymbol{G}^{F_\epsilon}$ for an appropriate Frobenius morphism $F_\epsilon : \widetilde{\boldsymbol{G}} \to \widetilde{\boldsymbol{G}}$.

2. Characters

2A. *Lusztig induction.* For this section, we let $\boldsymbol{H}$ be any connected reductive group over $\overline{\mathbb{F}}_q$ with Frobenius map F, and write $H = \boldsymbol{H}^F$. For any F-stable Levi subgroup $\boldsymbol{L}$ of $\boldsymbol{H}$, contained in a parabolic subgroup $\boldsymbol{P}$, we write $L = \boldsymbol{L}^F$ and denote by $R_L^H = R_{\boldsymbol{L}\subset\boldsymbol{P}}^{\boldsymbol{H}}$ the Lusztig (or twisted) induction functor. When $\boldsymbol{P}$ may be chosen to be an F-stable parabolic, then R_L^H becomes Harish-Chandra induction. When $\boldsymbol{L} = \boldsymbol{T}$ is chosen to be a maximal torus and θ is a character of $T = \boldsymbol{T}^F$, then $R_T^H(\theta)$ is the corresponding Deligne–Lusztig (virtual) character. We need the following basic result regarding actions on characters of finite reductive groups obtained through twisted induction.

Lemma 2.1. *Let $\boldsymbol{H}$ and $H = \boldsymbol{H}^F$ be as above. Let $\boldsymbol{L}$ be an F-stable Levi subgroup of $\boldsymbol{H}$, and write $L = \boldsymbol{L}^F$. Let χ be a character of L, $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and α a linear character of H which is trivial on unipotent elements. Then*

$$\sigma R_L^H(\chi) = R_L^H(\sigma\chi) \quad \text{and} \quad \alpha R_L^H(\chi) = R_L^H(\alpha\chi).$$

In particular, when $\boldsymbol{L} = \boldsymbol{T}$ is a maximal torus and $\chi = \theta$ is a character of $T = \boldsymbol{T}^F$, then we have

$$\sigma R_T^H(\theta) = R_T^H(\sigma\theta) \quad \text{and} \quad \alpha R_T^H(\theta) = R_T^H(\alpha\theta).$$

Proof. From [Digne and Michel 1991, Proposition 11.2], for any $g \in H$ we have

$$R_L^H(\chi)(g) = \frac{1}{|L|}\sum_{l\in L} \mathrm{Tr}((g, l^{-1})|X)\chi(l),$$

where $\mathrm{Tr}((g, l^{-1})|X)$ is the Lefschetz number corresponding to the $H \times L$-action on the ℓ-adic cohomology X of the relevant Deligne–Lusztig variety. In particular, these numbers are rational integers (by [Digne and Michel 1991, Corollary 10.6], for example). Thus,

$$\sigma R_L^H(\chi)(g) = \frac{1}{|L|}\sum_{l\in L} \mathrm{Tr}((g, l^{-1})|X)\sigma\chi(l) = R_L^H(\sigma\chi)(g).$$

The statement about α is [Digne and Michel 1991, Proposition 12.6(i)]. □

2B. ***Parametrization of characters of*** $\mathbf{GL}_n^\epsilon(q)$**.** We identify $\mathrm{GL}_1(\overline{\mathbb{F}}_q)$ with $\overline{\mathbb{F}}_q^\times$, and so F_ϵ acts on $\overline{\mathbb{F}}_q^\times$ via $F_\epsilon(a) = a^{\epsilon q}$. For any integer $k \geq 1$, we define T_k to be the multiplicative subgroup of $\overline{\mathbb{F}}_q^\times$ fixed by F_ϵ^k, that is,

$$T_k = (\overline{\mathbb{F}}_q^\times)^{F_\epsilon^k}.$$

We denote by $\widehat{T_k}$ the multiplicative group of complex-valued linear characters of T_k. Whenever $d \mid k$, we have the natural norm map $\mathrm{Nm}_{k,d} = \mathrm{Nm}$ from T_k to T_d, and the transpose map $\widehat{\mathrm{Nm}}$ gives a norm map from $\widehat{T_d}$ to $\widehat{T_k}$, where $\widehat{\mathrm{Nm}}(\xi) = \xi \circ \mathrm{Nm}$. We consider the direct limit of the character groups $\widehat{T_k}$ with respect to these norm maps, $\varinjlim \widehat{T_k}$, on which F_ϵ acts through its natural action on the groups T_k. Moreover, the fixed points of $\varinjlim \widehat{T_k}$ under F_ϵ^d can be identified with $\widehat{T_d}$. We let Θ denote the set of F_ϵ-orbits of $\varinjlim \widehat{T_k}$. The elements of Θ are sometimes called *simplices* (in [Green 1955; Turull 2001], for example). They are naturally dual objects to polynomials with roots given by an F_ϵ-orbit of $\overline{\mathbb{F}}_q^\times$.

For any orbit $\phi \in \Theta$, let $|\phi|$ denote the size of the orbit. Let $\mathcal{P}$ denote the set of all partitions of nonnegative integers, where we write $|\nu| = n$ if ν is a partition of n, and let $\mathcal{P}_n$ denote the set of all partitions of n. The irreducible characters of $\widetilde{G} = \mathrm{GL}_n^\epsilon(q)$ are parametrized by partition-valued functions on Θ. Specifically, given a function $\lambda : \Theta \to \mathcal{P}$, define $|\lambda|$ by

$$|\lambda| = \sum_{\phi \in \Theta} |\phi| \, |\lambda(\phi)|,$$

and define $\mathcal{F}_n$ by

$$\mathcal{F}_n = \{\lambda : \Theta \to \mathcal{P} \mid |\lambda| = n\}.$$

Then $\mathcal{F}_n$ gives a parametrization of the irreducible complex characters of $\widetilde{G}$. Given $\lambda \in \mathcal{F}_n$, we let $\widetilde{\chi}_\lambda$ denote the irreducible character corresponding to it.

We need several details regarding the structure of the character $\widetilde{\chi}_\lambda$. In the case $\epsilon = 1$, these facts all follow from the original work of Green [1955], and also appear from a slightly different point of view in [Macdonald 1995, Chapter IV]. For the case $\epsilon = -1$, the facts we need appear in [Thiem and Vinroot 2007], which contains relevant results from [Digne and Michel 1987; Lusztig and Srinivasan 1977].

First consider some $\lambda \in \mathcal{F}_n$ such that $\lambda(\phi)$ is a nonempty partition for exactly one $\phi \in \Theta$, and write $\widetilde{\chi}_\lambda = \widetilde{\chi}_{\lambda,\phi}$. Suppose that $|\phi| = d$, so that $|\lambda(\phi)| = n/d$. Then let $\omega^{\lambda(\phi)}$ be the irreducible character of the symmetric group $S_{n/d}$ parametrized by $\lambda(\phi) \in \mathcal{P}_{n/d}$. We fix this parametrization so that the partition $(1, 1, \ldots, 1)$ corresponds to the trivial character. For any $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_\ell) \in \mathcal{P}_{n/d}$, let $\omega^{\lambda(\phi)}(\gamma)$ denote the character $\omega^{\lambda(\phi)}$ evaluated at the conjugacy class parametrized by γ (where $(1, 1, \ldots, 1)$ corresponds to the identity), and let z_γ the size of the centralizer in $S_{n/d}$ of the class corresponding to γ. Let T_γ be the torus

$$T_\gamma = T_{d\gamma_1} \times T_{d\gamma_2} \times \cdots \times T_{d\gamma_\ell},$$

and let $\theta \in \phi$. Then we have

$$\widetilde{\chi}_{\lambda,\phi} = \pm \sum_{\gamma \in \mathcal{P}_{n/d}} \frac{\omega^{\lambda(\phi)}(\gamma)}{z_\gamma} R^{\widetilde{G}}_{T_\gamma}(\theta), \tag{1}$$

where the sign can be determined explicitly (see the remark after [Thiem and Vinroot 2007, Theorem 4.3], for example), but the sign will not have any impact for us. Note that from (1), it follows from our parametrization of characters of the symmetric group and [Digne and Michel 1991, Proposition 12.13] that the trivial character of $\widetilde{G}$ corresponds to $\lambda(\mathbf{1}) = (1, 1, \ldots, 1)$.

For an arbitrary $\lambda \in \mathcal{F}_n$, let $\phi_1, \phi_2, \ldots, \phi_r$ be precisely those elements in Θ such that $\lambda(\phi_i)$ is a nonempty partition, and let $d_i = |\phi_i|$. Let $n_i = d_i|\lambda(\phi_i)|$, and define L to be the Levi subgroup $L = \mathrm{GL}^\epsilon_{n_1}(q) \times \cdots \times \mathrm{GL}^\epsilon_{n_r}(q)$. The character $\widetilde{\chi}_\lambda$ is then given by

$$\widetilde{\chi}_\lambda = \pm R^{\widetilde{G}}_L(\widetilde{\chi}_{\lambda,\phi_1} \times \cdots \times \widetilde{\chi}_{\lambda,\phi_r}). \tag{2}$$

The sign only appears in the $\epsilon = -1$ case, and again can be determined explicitly. Note that (2) is Harish-Chandra induction in the case $\epsilon = 1$.

Remark 2.2. We may also describe the character $\widetilde{\chi}_\lambda$ in terms of Lusztig series and the Jordan decomposition of characters (see [Digne and Michel 1991, Chapter 13]) as follows. Each $\phi_i \in \Theta$ above corresponds to a polynomial with roots given by an F_ϵ-orbit of $\overline{\mathbb{F}}_q^\times$, as already mentioned. In this way, the collection

$$\{\phi_i \mid i = 1, \ldots, r\}$$

corresponds to a semisimple class (s) of $\widetilde{G}$ (we may identify $\widetilde{G}$ with its dual group in this case), and $\widetilde{\chi}_\lambda$ is an element of the Lusztig series $\mathcal{E}(\widetilde{G}, s)$. Then the centralizer in $\widetilde{G}$ of any element of (s) is of the form

$$C_{\widetilde{G}}(s) \cong \mathrm{GL}^{\epsilon_1}_{n_1/d_1}(q^{d_1}) \times \cdots \times \mathrm{GL}^{\epsilon_r}_{n_r/d_r}(q^{d_r}),$$

where $\epsilon_i = \epsilon^{d_i}$. The partition $\lambda(\phi_i)$ corresponds to a unipotent character $\psi_{\lambda(\phi_i)}$ of $\mathrm{GL}^{\epsilon_i}_{n_i/d_i}(q^{d_i})$. The character $\psi_\lambda = \otimes_{i=1}^r \psi_{\lambda(\phi_i)}$ is then a unipotent character of $C_{\widetilde{G}}(s)$, and the Jordan decomposition of $\widetilde{\chi}_\lambda$ is given by the $\widetilde{G}$-conjugacy class of pairs (s, ψ_λ).

2C. ***Restriction to $\mathrm{SL}_n^\epsilon(q)$ and actions on the parametrization.*** We now turn to the parametrization of the characters of $G = \mathrm{SL}_n^\epsilon(q)$ in terms of the characters of $\widetilde{G}$ described in the previous section. This is done for the case $\epsilon = 1$ in [Karkar and Green 1975; Lehrer 1973], and we adapt the methods there to handle the more general case of $\epsilon = \pm 1$.

Consider any Galois automorphism $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Through the natural action of σ on the character values of any $\theta \in \widehat{T}_d$, we have an action of σ on the orbits $\phi \in \Theta$. Given any $\lambda \in \mathcal{F}_n$, we define $\sigma\lambda$ by

$$\sigma\lambda(\phi) = \lambda(\sigma^{-1}\phi).$$

For $\alpha \in \widehat{T}_1$, define $\alpha\widetilde{\chi}$ and $\alpha\theta$ by the usual product of characters in $\mathrm{Irr}(\widetilde{G})$ and $\widehat{T}_d$, where we compose α with determinant and the norm maps, respectively. Then α acts on the orbits $\phi \in \Theta$ as well, and we get an action of α on $\mathcal{F}_n$ by defining $\alpha\lambda$ as

$$\alpha\lambda(\phi) = \lambda(\alpha^{-1}\phi).$$

We will need the following statements regarding these actions on the characters of $\widetilde{G}$.

Lemma 2.3. *Let $\lambda \in \mathcal{F}_n$. For any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and any $\alpha \in \widehat{T}_1$, we have*

$$\sigma\widetilde{\chi}_\lambda = \widetilde{\chi}_{\sigma\lambda} \quad \textit{and} \quad \alpha\widetilde{\chi}_\lambda = \widetilde{\chi}_{\alpha\lambda}.$$

Proof. We proceed in a manner similar to the proof of [Karkar and Green 1975, Proposition of Section 3], which proves this statement in the $\epsilon = 1$ case with the $\widehat{T}_1$ action. We begin by considering $\lambda \in \mathcal{F}_n$ such that $\lambda(\phi)$ is a nonempty partition for precisely one $\phi \in \Theta$, and so $\widetilde{\chi}_\lambda = \widetilde{\chi}_{\lambda,\phi}$ is given by (1). Given $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, since each $\omega^{\lambda(\phi)}(\gamma)$ and each z_γ is a rational integer, we have

$$\sigma\widetilde{\chi}_{\lambda,\phi}=\pm\sum_{\gamma\in\mathcal{P}_{n/d}}\frac{\omega^{\lambda(\phi)}(\gamma)}{z_\gamma}\sigma R^{\widetilde{G}}_{T_\gamma}(\theta)=\pm\sum_{\gamma\in\mathcal{P}_{n/d}}\frac{\omega^{\lambda(\phi)}(\gamma)}{z_\gamma}R^{\widetilde{G}}_{T_\gamma}(\sigma\theta),$$

by Lemma 2.1. Since $\sigma\theta\in\sigma\phi$ and $\sigma\lambda(\sigma\phi)=\lambda(\phi)$, then we have $\sigma\widetilde{\chi}_{\lambda,\phi}=\widetilde{\chi}_{\sigma\lambda,\sigma\phi}$. Similarly, if $\alpha\in\widehat{T}_1$, we have by Lemma 2.1

$$\alpha\widetilde{\chi}_{\lambda,\phi}=\pm\sum_{\gamma\in\mathcal{P}_{n/d}}\frac{\omega^{\lambda(\phi)}(\gamma)}{z_\gamma}\alpha R^{\widetilde{G}}_{T_\gamma}(\theta)=\pm\sum_{\gamma\in\mathcal{P}_{n/d}}\frac{\omega^{\lambda(\phi)}(\gamma)}{z_\gamma}R^{\widetilde{G}}_{T_\gamma}(\alpha\theta),$$

and since $\alpha\theta\in\alpha\phi$, we have $\alpha\widetilde{\chi}_{\lambda,\phi}=\widetilde{\chi}_{\alpha\lambda,\alpha\phi}$.

Now consider an arbitrary $\lambda\in\mathcal{F}_n$, with $\widetilde{\chi}_\lambda$ given by (2). By applying Lemma 2.1, along with the first case just proved, we have

$$\begin{aligned}\sigma\widetilde{\chi}_\lambda&=\pm R^{\widetilde{G}}_L(\sigma(\widetilde{\chi}_{\lambda,\phi_1}\times\cdots\times\widetilde{\chi}_{\lambda,\phi_r}))\\&=\pm R^{\widetilde{G}}_L(\sigma\widetilde{\chi}_{\lambda,\phi_1}\times\cdots\times\sigma\widetilde{\chi}_{\lambda,\phi_r})\\&=\pm R^{\widetilde{G}}_L(\widetilde{\chi}_{\sigma\lambda,\sigma\phi_1}\times\cdots\times\widetilde{\chi}_{\sigma\lambda,\sigma\phi_r})\\&=\widetilde{\chi}_{\sigma\lambda}.\end{aligned}$$

Similarly, if we replace σ with $\alpha\in\widehat{T}_1$, we have $\alpha\widetilde{\chi}_\lambda=\widetilde{\chi}_{\alpha\lambda}$ as claimed. □

Note that we may identify $\widetilde{G}/G$ with T_1, and directly from Clifford theory we know every character χ of G appears in some multiplicity-free restriction of a character $\widetilde{\chi}_\lambda$ of $\widetilde{G}$. The restrictions of two different irreducible characters of $\widetilde{G}$ are either equal, or have no irreducible constituents of $\mathrm{Irr}(G)$ in common. With this, the next result is all that is needed to parametrize $\mathrm{Irr}(G)$.

Lemma 2.4. *Let $\lambda,\mu\in\mathcal{F}_n$, with corresponding characters $\widetilde{\chi}_\lambda,\widetilde{\chi}_\mu\in\mathrm{Irr}(\widetilde{G})$. Then*

$$\mathrm{Res}^{\widetilde{G}}_G(\widetilde{\chi}_\lambda)=\mathrm{Res}^{\widetilde{G}}_G(\widetilde{\chi}_\mu)$$

if and only if there exists some $\alpha\in\widehat{T}_1$ such that $\lambda=\alpha\mu$.

Proof. This follows directly from [Karkar and Green 1975, Theorem 1(i)] and Lemma 2.3. □

Consider any irreducible character χ of G, so χ is a constituent of $\mathrm{Res}^{\widetilde{G}}_G(\widetilde{\chi}_\lambda)$ for some $\lambda\in\mathcal{F}_n$. That is, $\chi\in\mathrm{Irr}(G|\widetilde{\chi}_\lambda)$. The other constituents of this restriction are $\widetilde{G}$-conjugates of χ. Note that the field of values of χ is invariant under conjugation by $\widetilde{G}$, and so in studying this field of character values it is not important which constituent we choose. Since we will see below that $\mathbb{Q}(\chi)$ is closely related to the field of values of $\mathrm{Res}^{\widetilde{G}}_G(\widetilde{\chi}_\lambda)$, we give this a name.

Definition 1. We let $\mathbb{F}_\lambda:=\mathbb{Q}(\mathrm{Res}^{\widetilde{G}}_G(\widetilde{\chi}_\lambda))$ denote the field of values of $\mathrm{Res}^{\widetilde{G}}_G(\widetilde{\chi}_\lambda)$.

Further, given $\lambda \in \mathcal{F}_n$, we define the group $\mathcal{I}(\lambda)$ as

$$\mathcal{I}(\lambda) = \bigcap \{\ker \alpha \mid \alpha \in \widehat{T_1} \text{ such that } \alpha\lambda = \lambda\} \leq T_1.$$

We collect some basic properties of $\mathcal{I}(\lambda)$ in the following.

Proposition 2.5. *Let $\chi \in \operatorname{Irr}(G|\widetilde{\chi}_\lambda)$. Then*:

- *The stabilizer in $\widetilde{G}$ of χ is the set of elements with determinant in $\mathcal{I}(\lambda)$.*
- *The stabilizer of λ in $\widehat{T_1}$ is the set of elements whose kernel contains $\mathcal{I}(\lambda)$.*
- *The index $[T_1 : \mathcal{I}(\lambda)]$ divides $\gcd(q - \epsilon, n)$.*

Proof. The proof is exactly as in [Turull 2001, Propositions 4.2 and 4.3 and Corollary 4.4], using Clifford theory and Lemma 2.3. □

2D. ***Remarks on generalized Gelfand–Graev characters.*** We recall here some subgroups described in [Geck 2004, Section 2] used in the construction of the characters of generalized Gelfand–Graev representations (GGGRs). We introduce only the essentials for our purposes, and refer the reader to [Geck 2004; Kawanaka 1985; Taylor 2016], for example, for more details.

First, let $\boldsymbol{T}$ and $\boldsymbol{B}$ be an F_ϵ-stable maximal torus and Borel subgroup of $\boldsymbol{G}$, respectively, where $\boldsymbol{T} \leq \boldsymbol{B} = \boldsymbol{T}\boldsymbol{U}$, with unipotent radical $\boldsymbol{U}$. Let Φ be the root system of $\boldsymbol{G}$ with respect to $\boldsymbol{T}$ and $\Phi^+ \subset \Phi$ the set of positive roots determined by $\boldsymbol{B}$.

To each unipotent class $\mathcal{C}$ in $\boldsymbol{G}$ (or, equivalently, in $\widetilde{\boldsymbol{G}}$), there is associated a weighted Dynkin diagram $d : \Phi \to \mathbb{Z}$ and F_ϵ-stable groups

$$\boldsymbol{U}_{d,i} := \langle X_\alpha \mid \alpha \in \Phi^+,\ d(\alpha) \geq i \rangle \leq \boldsymbol{U},$$

where X_α denotes the root subgroup in $\boldsymbol{B}$ corresponding to α. In particular, $\boldsymbol{P}_d := N_{\widetilde{\boldsymbol{G}}}(\boldsymbol{U}_{d,1})$ is an F_ϵ-stable parabolic subgroup of $\widetilde{\boldsymbol{G}}$ and $\boldsymbol{U}_{d,i} \lhd \boldsymbol{P}_d$ for each $i = 1, 2, 3, \ldots$. We will further write $U_{d,i} := \boldsymbol{U}_{d,i}^{F_\epsilon}$ and $P_d := \boldsymbol{P}_d^{F_\epsilon}$.

Given $u \in \mathcal{C} \cap U_{d,2}$, the characters of GGGRs (which we will also refer to as GGGRs) of $\widetilde{G}$ (resp. G) are constructed by inducing certain linear characters $\varphi_u : U_{d,2} \to \mathbb{C}^\times$ to $\widetilde{G}$ (resp. G). In particular, the values of φ_u are all p-th roots of unity. Strictly speaking, the GGGRs are actually rational multiples of the induced character:

$$\widetilde{\Gamma}_u = [U_{d,1} : U_{d,2}]^{-\frac{1}{2}} \operatorname{Ind}_{U_{d,2}}^{\widetilde{G}}(\varphi_u) \quad \text{and} \quad \Gamma_u = [U_{d,1} : U_{d,2}]^{-\frac{1}{2}} \operatorname{Ind}_{U_{d,2}}^{G}(\varphi_u).$$

The following is [Schaeffer Fry and Taylor 2018, Proposition 10.11], which is a consequence of [Tiep and Zalesskii 2004, Theorem 1.8, Lemma 2.6, and Theorem 10.10].

Proposition 2.6. *Let Γ_u be a GGGR of G. Then the following hold.*

(1) *If q is a square, or n is odd, or $n/(n, q-\epsilon)$ is even, then the values of Γ_u are integers.*

(2) *Otherwise, the values of Γ_u lie in $\mathbb{Q}(\sqrt{\eta p})$, where $\eta \in \{\pm 1\}$ is such that $p \equiv \eta \bmod 4$.*

Note that case (1) of Proposition 2.6 includes the case that q is even.

3. Initial results on fields of values

Keep the notation from above, so that $G = \mathrm{SL}_n^\epsilon(q)$, $\widetilde{G} = \mathrm{GL}_n^\epsilon(q)$, and the characters of $\widetilde{G}$ are denoted by $\widetilde{\chi}_\lambda$ for $\lambda \in \mathcal{F}_n$. For $\lambda \in \mathcal{F}_n$, let $\mathbb{Q}(\lambda)$ denote the field obtained from $\mathbb{Q}$ by adjoining the values of the characters in the orbits $\phi \in \Theta$ such that $\lambda(\phi)$ is nonempty.

We define $\mathrm{Galg}(\lambda)$ and $\mathrm{Galr}(\lambda)$ as in [Turull 2001]. That is, $\mathrm{Galg}(\lambda)$ is the stabilizer of λ in $\mathrm{Gal}(\mathbb{Q}(\lambda)/\mathbb{Q})$ and

$$\mathrm{Galr}(\lambda) = \{\sigma \in \mathrm{Gal}(\mathbb{Q}(\lambda)/\mathbb{Q}) \mid \sigma\lambda = \alpha\lambda \text{ for some } \alpha \in \widehat{T_1}\}.$$

Theorem 3.1. *Let $\lambda \in \mathcal{F}_n$. Then $\mathbb{Q}(\widetilde{\chi}_\lambda) = \mathbb{Q}(\lambda)^{\mathrm{Galg}(\lambda)}$ and $\mathbb{F}_\lambda = \mathbb{Q}(\lambda)^{\mathrm{Galr}(\lambda)}$ (recall Definition 1). That is, the field of values for $\widetilde{\chi}_\lambda$ and its restriction to G are the fixed fields of* $\mathrm{Galg}(\lambda)$ *and* $\mathrm{Galr}(\lambda)$, *respectively.*

Proof. Given Lemma 2.3, the proof is exactly the same as that of [Turull 2001, Propositions 2.8 and 3.4]. □

Note that since the members of $\phi \in \Theta$ are characters of T_d for some d, it follows that $\mathbb{Q}(\lambda) = \mathbb{Q}(\zeta_m)$ is the field obtained from $\mathbb{Q}$ by adjoining some primitive m-th root of unity ζ_m, where $\gcd(m, p) = 1$. Since $\mathbb{F}_\lambda \subseteq \mathbb{Q}(\lambda)$, it follows that $\mathbb{F}_\lambda \cap \mathbb{Q}(\zeta_p) = \mathbb{Q}$ for any primitive p-th root of unity ζ_p.

Remark 3.2. As in the proof of [Turull 2001, Proposition 6.2], the Galois automorphism $\sigma_{-1} : \mathbb{Q}(\zeta_m) \to \mathbb{Q}(\zeta_m)$ satisfying $\sigma_{-1}(\zeta_m) = \zeta_m^{-1}$ induces complex conjugation on $\mathbb{Q}(\lambda)$. Hence $\widetilde{\chi}_\lambda$ (resp. $\mathrm{Res}_G^{\widetilde{G}}(\widetilde{\chi}_\lambda)$) is real-valued if and only if $\sigma_{-1} \in \mathrm{Galg}(\lambda)$ (resp. $\sigma_{-1} \in \mathrm{Galr}(\lambda)$).

Proposition 3.3. *Let $\chi \in \mathrm{Irr}(G|\widetilde{\chi}_\lambda)$. Keep the notation above. Then:*

(1) *If q is square, or n is odd, or $n/(n, q-\epsilon)$ is even, then $\mathbb{Q}(\chi) = \mathbb{F}_\lambda$.*

(2) *Otherwise, $\mathbb{F}_\lambda \subseteq \mathbb{Q}(\chi) \subseteq \mathbb{F}_\lambda(\sqrt{\eta p})$, where $\eta \in \{\pm 1\}$ is such that $p \equiv \eta \bmod 4$.*

In particular, χ is real-valued if and only if $\mathrm{Res}_G^{\widetilde{G}}(\widetilde{\chi}_\lambda)$ is, except possibly when $q \equiv 3 \bmod 4$ and $2 \leq n_2 \leq (q-\epsilon)_2$.

Proof. Write $\mathbb{F} := \mathbb{F}_\lambda$ and $\widetilde{\chi} := \widetilde{\chi}_\lambda$. First, we remark that certainly $\mathbb{F} \subseteq \mathbb{Q}(\chi)$, by its definition, since $\mathrm{Res}_G^{\widetilde{G}}(\widetilde{\chi})$ is the sum of $\widetilde{G}$-conjugates of χ.

Let $\widetilde{\Gamma}$ be a GGGR of $\widetilde{G}$ such that $\langle \widetilde{\Gamma}, \widetilde{\chi} \rangle_{\widetilde{G}} = 1$, which exists by a well-known result of Kawanaka (see [Kawanaka 1985, 3.2.18] for large p or [Taylor 2016, 15.7]).

Further, there exists a GGGR, Γ, of G such that $\widetilde{\Gamma} = \mathrm{Ind}_G^{\widetilde{G}}(\Gamma)$. Then Frobenius reciprocity yields that there is a unique irreducible constituent $\chi_0 \in \mathrm{Irr}(G|\widetilde{\chi})$ satisfying $\langle \Gamma, \chi_0 \rangle_G = 1$. Without loss, we may assume χ is this χ_0, as the field of values is invariant under $\widetilde{G}$-conjugation.

Write $\mathbb{K} = \mathbb{F}(\sqrt{\eta p})$. Let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{F})$ in case (1), and let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{K})$ in case (2). Then by Proposition 2.6, $\sigma\chi$ is also a constituent of $\Gamma = \sigma\Gamma$ occurring with multiplicity 1. As $\mathrm{Res}_G^{\widetilde{G}}(\widetilde{\chi})$ is invariant under σ, however, we have that $\sigma\chi$ is also a constituent of the restriction $\mathrm{Res}_G^{\widetilde{G}}(\widetilde{\chi})$. Hence we see that $\sigma\chi = \chi$, by uniqueness, and hence $\mathbb{Q}(\chi) \subseteq \mathbb{F}$ in case (1) and $\mathbb{Q}(\chi) \subseteq \mathbb{K}$ in case (2). □

In our main results below when q is odd, characters of T_1 of 2-power order will play an important role. In particular, we denote by sgn the unique member of $\widehat{T}_1$ of order 2.

Lemma 3.4. *Let q be odd and let $\chi \in \mathrm{Irr}(G|\widetilde{\chi}_\lambda)$ and write $I := \mathrm{stab}_{\widetilde{G}}(\chi)$. Then $[\widetilde{G} : I]$ is even if and only if $\mathrm{sgn}\lambda = \lambda$.*

Proof. Note that 2 divides $[\widetilde{G} : I]$ if and only if $[I : G]_2 \leq \frac{1}{2}(q-\epsilon)_2$, if and only if $\mathcal{I}(\lambda)$ is contained in the unique subgroup of $\widetilde{G}/G$ of order $\frac{1}{2}(q-\epsilon)$. But notice that this is exactly the kernel of sgn as an element of $\widehat{T}_1$. □

Lemma 3.5. *Let q be odd and let $\chi \in \mathrm{Irr}(G|\widetilde{\chi}_\lambda)$. If $\mathrm{sgn}\lambda \neq \lambda$, then $\mathbb{F}_\lambda = \mathbb{Q}(\chi)$.*

Proof. Write $\mathbb{F} := \mathbb{F}_\lambda$ and recall that $\mathbb{F} \subseteq \mathbb{Q}(\chi) \subseteq \mathbb{F}(\sqrt{\eta p})$. Let $\mathbb{K} = \mathbb{F}(\sqrt{\eta p})$, so that $\mathbb{K}$ is a quadratic extension of $\mathbb{F}$. Let τ be the generator of $\mathrm{Gal}(\mathbb{K}/\mathbb{F})$ and write I for the stabilizer of χ under $\widetilde{G}$. Then note that τ^2 necessarily fixes χ, and by definition τ fixes $\mathrm{Res}_G^{\widetilde{G}}(\widetilde{\chi})$, which by Clifford theory is the sum of the $[\widetilde{G} : I]$ conjugates of χ under the action of $\widetilde{G}$.

We prove the contrapositive. Suppose $\mathbb{F} \neq \mathbb{Q}(\chi)$, so that τ does not fix χ. Then since the field of values is invariant under $\widetilde{G}$-conjugation, it follows that the orbit of χ under $\widetilde{G}$ can be partitioned into pairs conjugate to $\{\chi, \tau\chi\}$. Hence the size of the orbit, $[\widetilde{G} : I]$, must be even, so $\mathrm{sgn}\,\lambda = \lambda$ by Lemma 3.4. □

4. Unipotent elements

To deal with the remaining cases (in particular, when $q \equiv \eta \bmod 4$ is nonsquare, $\eta \in \{\pm 1\}$, $\epsilon = -1$, and $2 \leq n_2 \leq (q+1)_2$), we will continue to employ the use of GGGRs. For this, we will need to analyze certain aspects of conjugacy of unipotent elements. Here, the observations in [Schaeffer Fry and Vinroot 2016] on this subject will be useful.

In particular, if a unipotent element of $\widetilde{G} = \mathrm{GL}_n^\epsilon(q)$ for $\epsilon \in \{\pm 1\}$ has m_k Jordan blocks of size k (that is, m_k elementary divisors of the form $(t-1)^k$), then we may find a conjugate in $\widetilde{G}$ of the form $\bigoplus_k \widetilde{J}_k^{m_k}$, where the sum is over only those k such that $m_k \neq 0$ and each $\widetilde{J}_k \in \mathrm{GL}_k^\epsilon(q)$ is a Jordan block of size k. The following lemma will be useful throughout the section.

Lemma 4.1 [Schaeffer Fry and Vinroot 2016, Lemma 3.2]. *Let u be a unipotent element in $\widetilde{G}$ with m_k Jordan blocks of size k for each $1 \leq k \leq n$. For each k such that $m_k \neq 0$, let $\delta_k \in T_1$ be arbitrary. Then there exists some $g \in C_{\widetilde{G}}(u)$ such that* $\det(g) = \prod_k \delta_k^k$.

We now turn our attention to the case $\epsilon = -1$ for the remainder of this section. Let ζ_p be a primitive p-th root of unity in $\mathbb{C}$. In what follows, we let b be a fixed integer such that $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ is generated by the map $\tau : \zeta_p \mapsto \zeta_p^b$. Note that $(b, p) = 1$ and b has multiplicative order $p-1$ modulo p. Further, note that τ also induces the map $\sqrt{\eta p} \mapsto -\sqrt{\eta p}$ generating $\mathrm{Gal}(\mathbb{Q}(\sqrt{\eta p})/\mathbb{Q})$. Let $\bar{b}$ denote the image of b under a fixed isomorphism $(\mathbb{Z}/p\mathbb{Z})^\times \to \mathbb{F}_p^\times$, so that $\bar{b}$ generates $\mathbb{F}_p^\times$.

Note that by [Tiep and Zalesskii 2004, Theorem 1.9], every unipotent element u of $\mathrm{GU}_n(q)$ is conjugate to u^b in $C_{\widetilde{G}}(s)$, where s is a semisimple element in $C_{\widetilde{G}}(u)$. We are interested in making precise statements about such a conjugating element.

Let u be a regular unipotent element of $\mathrm{GU}_n(q)$, identified as in [Schaeffer Fry and Vinroot 2016, Lemma 5.1]. Employing the same argument as is used there, we see that an element conjugating u to u^b must have diagonal

$$(\bar{b}^{n-1}\beta, \bar{b}^{n-2}\beta, \ldots, \bar{b}\beta, \beta),$$

where $\beta \in \mathbb{F}_{q^2}^\times$ and $\bar{b}^{n-1}\beta^{q+1} = 1$. Note the determinant of such an element is $\bar{b}^{\binom{n}{2}}\beta^n$ and that the condition that $\bar{b}^{n-1}\beta^{q+1} = 1$ yields that β^{q+1} is a $(p-1)$-root of unity.

Lemma 4.2. *Let $q \equiv \eta \bmod 4$ be nonsquare with $\eta \in \{\pm 1\}$ and let u be a regular unipotent element of* $\mathrm{GU}_n(q)$. *Keep the notation above. Then*:

(a) *If n is even, then β_2 is a primitive $(q^2-1)_2$-root of unity in $\mathbb{F}_{q^2}^\times$.*

(b) *There is an element x in $\mathrm{GU}_n(q)$ such that $u^x = u^b$ and $|\det(x)|$ is a 2-power.*

(c) *If $n \not\equiv 0 \bmod 4$, there is an element x in $\mathrm{GU}_n(q)$ such that $u^x = u^b$ and $|\det(x)| = (q+1)_2$. Further*:

  (i) *If $n \equiv 2 \bmod 4$, then $|\det(x)|_2 = (q+1)_2$ for any $x \in \mathrm{GU}_n(q)$ satisfying $u^x = u^b$.*

  (ii) *If n is odd, there is also an element $x' \in \mathrm{SU}_n(q)$ such that $u^{x'} = u^b$.*

(d) *If $n \equiv 0 \bmod 4$, there is no element x in $\mathrm{GU}_n(q)$ such that $u^x = u^b$ and $|\det(x)| = (q+1)_2$.*

Proof. For part (a) note that $|\bar{b}^{n-1}|$ has the same 2-part as $|\bar{b}|$ since n is even, so $|\beta_2^{q+1}| = (p-1)_2 = (q-1)_2$ since q is nonsquare. Hence the multiplicative order of β_2 is $2(q-\eta)_2 = (q^2-1)_2$.

To prove the rest, we begin by showing that if $n \not\equiv 0 \mod 4$, then we may find an x in $\mathrm{GU}_n(q)$ such that $u^x = u^b$ and $|\det(x)|_2 = (q+1)_2$, and that this is impossible for $n \equiv 0 \mod 4$. Let $x \in \mathrm{GU}_n(q)$ satisfy $u^x = u^b$, so that x is of the form discussed before the statement of the lemma.

If $n \equiv 2 \mod 4$, then β_2 is a primitive $(q^2-1)_2$-root of unity by (a), and hence β_2^n is a $(q-\eta)_2$-root of unity. Here $\bar{b}_2^{\binom{n}{2}}$ has the same order as $\bar{b}_2$, since $\binom{n}{2}$ is odd. If $\eta = -1$, then since $|\bar{b}|_2 = 2$, we see $\bar{b}_2^{\binom{n}{2}}\beta_2^n = \pm\beta_2^n$. Then any $x \in \mathrm{GU}_n(q)$ satisfying $u^x = u^b$ must satisfy $|\det(x)|_2 = (q+1)_2$ in this case. If $\eta = 1$, note that $(q+1)_2 = 2$, so we must just show that $\det(x)$ has even order. Here $|\bar{b}|_2 = (q-1)_2$, and $\bar{b}^{\binom{n}{2}} \cdot \beta^n$ has even order unless $\bar{b}_2^{\binom{n}{2}} \cdot \beta_2^n = 1$. However, the latter situation, combined with the fact that $\bar{b}_2^{n-1}\beta_2^{q+1} = 1$, yields that $\beta_2^{-n} = (\bar{b}_2^{n-1})^{n/2} = (\beta_2^{-1-q})^{n/2}$, and hence $\beta_2^{n((q+1)/2-1)} = 1$. But since $n_2 = 2$, this means that $\beta_2^{q-1} = 1$, contradicting that β_2 has order $(q^2-1)_2$. Then it must be that $\det(x)$ has even order, and we again see that $|\det(x)|_2 = (q+1)_2$ in this case.

Now assume n is odd and let $\tilde{x}$ be an element in $\mathrm{GU}_n(q)$ satisfying $u^{\tilde{x}} = u^b$. Then certainly $\det(\tilde{x}) \in T_1$, so we may use Lemma 4.1 to replace $\tilde{x}$ with some $x \in \mathrm{GU}_n(q)$ satisfying $\det(x) = \det(\tilde{x}) \cdot \delta^n$ for any $\delta \in T_1$. In particular, note that $|\delta^n| = |\delta|$ for any $(q+1)_2$-root of unity δ, since n is odd. Then we may choose δ so that $\det(\tilde{x})_2\delta^n$ is a primitive $(q+1)_2$-root of unity, yielding $|\det(x)|_2 = (q+1)_2$. Alternatively, we may choose δ so that $\det(\tilde{x})_2\delta^n = 1$, yielding $|\det(x)|_2 = 1$.

Finally, let $n \equiv 0 \mod 4$. Then note that neither $\bar{b}_2^{\binom{n}{2}}\beta_2^n$ nor δ^n for any $\delta \in T_1$ can be a primitive $(q+1)_2$-root of unity, so $|\det(x)|_2 < (q+1)_2$.

It remains to show that in all cases, x can be chosen such that $|\det(x)|_{2'} = 1$. Since β^{q+1} is a $(p-1)$-root of unity, we may decompose the determinant of x into $\beta_2^n \cdot \beta_{(q+1)_{2'}}^n \cdot y$, where y is a $(p-1)$-root of unity. However, we also know that the determinant is a $(q+1)$-root and an odd prime cannot divide both $p-1$ and $q+1$. Hence y must be a 2-power root of unity, and we may replace x with an element of determinant $\beta_2^n \cdot y$, using Lemma 4.1. □

Although there is no x satisfying the conclusion of Lemma 4.2(c) if u is a regular unipotent element when 4 divides n, we can generalize to the following statement about more general unipotent elements when $4 \nmid n$.

Corollary 4.3. *Let $q \equiv \eta \mod 4$ be nonsquare with $\eta \in \{\pm 1\}$. If u is a unipotent element of $\mathrm{GU}_n(q)$ satisfying at least one of the following:*

(1) *u has an odd number of elementary divisors of the form $(t-1)^k$ with $k \equiv 2 \mod 4$;*

(2) *u has an elementary divisor of the form $(t-1)^k$ with k odd,*

then u is conjugate to u^b by an element x satisfying $|\det(x)| = (q+1)_2$.

In particular, if n is not divisible by 4, any unipotent element is conjugate to u^b by an element x satisfying $|\det(x)| = (q+1)_2$.

Proof. Indeed, viewing u as $\bigoplus_k \widetilde{J}_k^{m_k}$ as in [Schaeffer Fry and Vinroot 2016, Section 3.2], we may find elements x_k for each $1 \leq k \leq n$ as in Lemma 4.2 conjugating each $\tilde{J}_k$ to $\tilde{J}_k^b$. In case (1), we see that the product $\bigoplus_k {x_k}^{m_k}$ will satisfy the statement, after possibly replacing x_k with x_k' as in Lemma 4.2(c)(ii) for any odd k, so $|\det(x_k')| = 1$.

If (2) holds, but (1) does not hold, $y = \bigoplus_{2|k} {x_k}^{m_k}$ will satisfy $|\det(y)| = |\det(y)|_2 < (q+1)_2$, since the product of an even number of primitive $(q+1)_2$-roots of unity will no longer be primitive. We may use Lemma 4.2 to obtain x_k for some k odd such that $|\det(x_k)| = (q+1)_2$, and replace the remaining x_k for odd k with an element x_k' satisfying $|\det(x_k')| = 1$. The resulting $\bigoplus_k {x_k}^{m_k}$ will satisfy the statement.

The last statement follows, since if n is odd, we must be in case (2), and if $n \equiv 2 \bmod 4$, we must be in case (1) or (2). □

Remark 4.4. At least one of conditions (1) and (2) of Corollary 4.3 must occur if $n \equiv 2 \bmod 4$, and condition (1) implies condition (2) if $n \equiv 0 \bmod 4$. Further, when $\eta = 1$, the condition $n_2 \leq (q+1)_2$ induced from Proposition 3.3 yields that $n \equiv 2 \bmod 4$.

We now address the case that 4 divides n, $q \equiv 3 \bmod 4$, and that neither of the conditions in Corollary 4.3 occur.

Lemma 4.5. *Let $q \equiv 3 \bmod 4$ and let $n \equiv 0 \bmod 4$ such that $n_2 \leq (q+1)_2$. Let u be a unipotent element of $\mathrm{GU}_n(q)$ with no elementary divisors $(t-1)^k$ with k odd. Then u is conjugate to u^b by an element x satisfying $|\det(x)| = (q^2-1)_2/n_2$.*

Proof. As in the proof of Corollary 4.3, let $\tilde{x} = \bigoplus_k {x_k}^{m_k}$, where for each k such that $m_k \neq 0$, x_k is an element of $\mathrm{GU}_k(q)$ conjugating $\widetilde{J}_k$ to $\widetilde{J}_k^b$ as in Lemma 4.2. Now, each x_k has determinant $\pm(\beta_k)_2{}^k$, where $(\beta_k)_2$ is a primitive $(q^2-1)_2$-root of unity in $\mathbb{F}_{q^2}^\times$, by Lemma 4.2, since the y found there has multiplicative order $(p-1)_2 = 2$. Then taking $\delta_k \in \mathbb{F}_{q^2}^\times$ to be the primitive $(q+1)_2$-root of unity $\delta_k = (\beta_k)_2^2$, we may use Lemma 4.1 to replace x_k with an element whose determinant is $\pm(\beta_k)_2^k \delta_k^{rk} = \pm(\beta_k)_2^{k(2r+1)}$ for any odd r, yielding that we may replace each x_k with an element whose determinant is $\pm\beta_2^k$ for a fixed primitive $(q^2-1)_2$-root of unity β_2. Hence the resulting x satisfies $\det(x) = \pm\beta_2^n$, which has the stated order. □

5. Application to GGGRs

Here we keep the notation of Section 2D and return to the more general case that $\widetilde{G} = \mathrm{GL}_n^\epsilon(q)$ and $G = \mathrm{SL}_n^\epsilon(q)$ for $\epsilon \in \{\pm 1\}$. Let $\mathcal{C}$ be a fixed unipotent class of $\widetilde{G}$.

Lemma 5.1. *Let $u \in \mathcal{C} \cap U_{d,2}$ and suppose that x is an element normalizing $U_{d,2}$ and conjugating u to u^b. Then $\varphi_u^x = \varphi_u^b$.*

Proof. This follows from the construction of φ_u in [Taylor 2016, Section 5] or [Geck 2004, Section 2]. Indeed, for each g in $U_{d,2}$, we have $\varphi_u^x(g) = \varphi_u(xgx^{-1}) = \varphi_{x^{-1}ux}(g) = \varphi_{u^x}(g) = \varphi_{u^b}(g) = \varphi_u(g)^b$, where the second equality is noted in [Geck 2004, Remark 2.2]. □

Lemma 5.2. *Let $u \in \mathcal{C} \cap U_{d,2}$ and $\epsilon = -1$. Then the elements x found in Corollary 4.3 and Lemma 4.5 are members of P_d, and hence normalize $U_{d,2}$.*

Proof. First, note that $C_{\widetilde{\boldsymbol{G}}}(u) \leq \boldsymbol{P}_d$. Indeed, this is noted in [Kawanaka 1986, Theorem 2.1.1] for simply connected groups, and here we have $\widetilde{\boldsymbol{G}} = \boldsymbol{G}Z(\widetilde{\boldsymbol{G}})$ with $\boldsymbol{G}$ simply connected. Further, u is conjugate to u^b in $\boldsymbol{P}_d$ by [Schaeffer Fry and Taylor 2018, Lemma 4.6]. So $u^x = u^b = u^y$ for some $y \in \boldsymbol{P}_d$, which yields that $xy^{-1} \in C_{\widetilde{\boldsymbol{G}}}(u)$, and hence $x \in \widetilde{G} \cap \boldsymbol{P}_d$. This shows that x is contained in P_d, which contains $U_{d,2}$ as a normal subgroup. □

Lemma 5.3. *Let $G = \mathrm{SL}_n^\epsilon(q)$ and $\widetilde{G} = \mathrm{GL}_n^\epsilon(q)$, with $\epsilon \in \{\pm 1\}$. Let $\widetilde{\chi} := \widetilde{\chi}_\lambda \in \mathrm{Irr}(\widetilde{G})$ and let*

$$\widetilde{\Gamma}_u = [U_{1,d} : U_{2,d}]^{-\frac{1}{2}} \mathrm{Ind}_{U_{d,2}}^{\widetilde{G}}(\varphi_u)$$

be a generalized Gelfand–Graev character of $\widetilde{G}$ such that $\langle \widetilde{\Gamma}_u, \widetilde{\chi} \rangle_{\widetilde{G}} = 1$. Let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{F}_\lambda)$ and let $x \in \widetilde{G}$ normalizing $U_{d,2}$ such that $\sigma\varphi_u = \varphi_u^x$. Then for $\chi \in \mathrm{Irr}(G|\widetilde{\chi})$, there is some conjugate χ_0 of χ such that $\sigma\chi_0 = \chi_0^x$.

Proof. Let Γ_u be such that $\widetilde{\Gamma}_u = \mathrm{Ind}_G^{\widetilde{G}} \Gamma_u$ and $\Gamma_u = r \cdot \mathrm{Ind}_{U_{d,2}}^G(\varphi_u)$ where $r = [U_{1,d} : U_{2,d}]^{-1/2}$. Then by Clifford theory and Frobenius reciprocity, there is a unique conjugate, χ_0, of χ such that $\chi_0 \in \mathrm{Irr}(G|\widetilde{\chi})$ and $\langle \Gamma_u, \chi_0 \rangle_G = 1$. Since $\mathrm{Res}_G^{\widetilde{G}}(\widetilde{\chi})$ is fixed by σ, we also see $\sigma\chi_0$ is the unique member of $\mathrm{Irr}(G|\widetilde{\chi})$ satisfying $\langle \sigma\Gamma_u, \sigma\chi_0 \rangle_G = 1$. But note that

$$\sigma\Gamma_u = r \cdot \mathrm{Ind}_{U_{d,2}}^G(\sigma\varphi_u) = r \cdot \mathrm{Ind}_{U_{d,2}}^G(\varphi_u^x) = \Gamma_u^x.$$

Then $\langle \sigma\Gamma_u, \chi_0^x \rangle_G = \langle \Gamma_u^x, \chi_0^x \rangle_G = 1$, forcing $\chi_0^x = \sigma\chi_0$ by uniqueness, since $\chi_0^x \in \mathrm{Irr}(G|\widetilde{\chi})$. □

6. Main results

Our first main result is an extension of [Turull 2001, Theorem 4.8] to the case of $G = \mathrm{SU}_n(q)$, describing the field of values $\mathbb{Q}(\chi)$ for each $\chi \in \mathrm{Irr}(G)$. Recall that we write $\mathbb{F}_\lambda$ for the field of values of the restriction to G of $\widetilde{\chi}_\lambda$ (see Definition 1).

Theorem 6.1. *Let $G = \mathrm{SL}_n^\epsilon(q)$ and $\widetilde{G} = \mathrm{GL}_n^\epsilon(q)$, with $\epsilon \in \{\pm 1\}$. Let $\lambda \in \mathcal{F}_n$ and let $\chi \in \mathrm{Irr}(G|\widetilde{\chi}_\lambda)$. Then $\mathbb{Q}(\chi) = \mathbb{F}_\lambda$ unless all of the following hold*:

- *p is odd.*
- *q is not square.*

- $2 \le n_2 \le (q-\epsilon)_2$.
- $\alpha\lambda = \lambda$ *for any element* $\alpha \in \widehat{T}_1$ *of order* n_2.

In the latter case, $\mathbb{Q}(\chi) = \mathbb{F}_\lambda(\sqrt{\eta p})$, *where* $\eta \in \{\pm 1\}$ *and* $p \equiv \eta \mod 4$.

Remark 6.2. In terms of the Jordan decomposition of characters described in Remark 2.2, the last condition in Theorem 6.1 may be translated as follows. Let $\widetilde{\chi}_\lambda$ have Jordan decomposition (s, ψ_λ), and for any $\alpha \in \widehat{T}_1$ of order n_2 in the condition, let $a \in T_1$ correspond to α through some fixed isomorphism. Then the condition is equivalent to the pair (s, ψ_λ) being $\widetilde{G}$-conjugate to the pair $(as, \psi_{\alpha\lambda})$.

Taking into consideration Remark 3.2, Theorem 6.1 immediately yields the following extension of [Turull 2001, Proposition 6.2].

Corollary 6.3. *Let* $G = \mathrm{SL}_n^\epsilon(q)$ *and* $\widetilde{G} = \mathrm{GL}_n^\epsilon(q)$, *with* $\epsilon \in \{\pm 1\}$. *Let* $\lambda \in \mathcal{F}_n$ *and let* $\chi \in \mathrm{Irr}(G|\widetilde{\chi}_\lambda)$. *Then the following are equivalent*:

- χ *is real-valued.*
- *There exists some* $\alpha' \in \widehat{T}_1$ *such that* $\sigma_{-1}\lambda = \alpha'\lambda$, *and if* p *is odd*, q *is not a square*, $2 \le n_2 \le (q-\epsilon)_2$, *and* $\alpha\lambda = \lambda$ *for any element* $\alpha \in \widehat{T}_1$ *of order* n_2, *then* $p \equiv 1 \mod 4$.

The remainder of this section will be devoted to proving Theorem 6.1 when $\epsilon = -1$. (Recall that Theorem 6.1 in the case $\epsilon = 1$ is [Turull 2001, Theorem 4.8].) We begin with an observation that holds in either case $\epsilon \in \{\pm 1\}$, but in particular restricts the situation of Corollary 4.3.

Proposition 6.4. *Let* $G = \mathrm{SL}_n^\epsilon(q)$ *and* $\widetilde{G} = \mathrm{GL}_n^\epsilon(q)$, *with* $\epsilon \in \{\pm 1\}$. *Let* $\widetilde{\chi} \in \mathrm{Irr}(\widetilde{G})$ *and let* $\widetilde{\Gamma}_u$ *be a GGGR of* $\widetilde{G}$ *such that* $\langle \widetilde{\Gamma}_u, \widetilde{\chi}\rangle_{\widetilde{G}} = 1$. *Further, assume that* u *has an elementary divisor of the form* $(t-1)^k$ *with* k *odd. Then* $[\widetilde{G} : I]$ *is odd, where* $I = \mathrm{stab}_{\widetilde{G}}(\chi)$ *for any* $\chi \in \mathrm{Irr}(G|\widetilde{\chi})$. *In particular, in this case*, $\mathbb{F}_\lambda = \mathbb{Q}(\chi)$ *by Lemma 3.5.*

Proof. Write $\widetilde{\Gamma}_u = [U_{1,d} : U_{2,d}]^{-1/2}\mathrm{Ind}_{U_{d,2}}^{\widetilde{G}}(\varphi_u)$. By Lemma 4.1, there is some $x \in C_{\widetilde{G}}(u)$ with determinant δ^k, where δ is a primitive $(q-\epsilon)_2$-root of unity in T_1. In particular, $|\det(x)| = (q-\epsilon)_2$ since k is odd, and $\varphi_u^x = \varphi_u$ since x normalizes $U_{d,2}$ as in the proof of Lemma 5.1. Then applying Lemma 5.3 with σ trivial yields that some conjugate χ_0 of χ satisfies $\chi_0^x = \chi_0$. This implies $[I : G]$ is divisible by $(q-\epsilon)_2$, so that $[\widetilde{G} : I]$ must be odd. □

For the remainder of this section, we will consider the case $\epsilon = -1$, so that $\widetilde{G} = \mathrm{GU}_n(q)$ and $G = \mathrm{SU}_n(q)$. In particular, Remark 4.4 and Proposition 6.4 yield that when $[\widetilde{G} : I]$ is even and n is divisible by 4, then neither condition in Corollary 4.3 holds. Further, if $[\widetilde{G} : I]$ is even and $n \equiv 2 \mod 4$, then Corollary 4.3(1) holds.

Proposition 6.5. *Let* $\epsilon = -1$ *and suppose that* $q \equiv \eta \bmod 4$ *is nonsquare with* $\eta \in \{\pm 1\}$ *and that* $n \equiv 2 \bmod 4$*. Then the converse of Lemma 3.5 holds. That is, for* $\chi \in \mathrm{Irr}(G|\widetilde{\chi}_\lambda)$, $\mathbb{F}_\lambda = \mathbb{Q}(\chi)$ *if and only if* $\mathrm{sgn}\,\lambda \neq \lambda$*. Alternatively,* $\mathbb{F}_\lambda(\sqrt{\eta p}) = \mathbb{Q}(\chi)$ *if and only if* $\mathrm{sgn}\,\lambda = \lambda$*.*

Proof. We must show that if $\mathrm{sgn}\,\lambda = \lambda$, then $\mathbb{F}_\lambda \neq \mathbb{Q}(\chi)$. First, recall that this condition on λ is equivalent to the condition that $[\widetilde{G} : I] = [T_1 : \mathcal{I}(\lambda)]$ is even, by Lemma 3.4. Since $n_2 = 2$, Proposition 2.5 yields that $[\widetilde{G} : I]_2 = 2$. This means that no $\widetilde{G}$-conjugate of χ can be fixed by any $\tilde{g} \in \widetilde{G}$ whose determinant satisfies $|\det(\tilde{g})|_2 = (q+1)_2$.

Abusing notation, we let τ also denote the unique element of $\mathrm{Gal}(\mathbb{F}_\lambda(\zeta_p)/\mathbb{F}_\lambda)$ that restricts to our fixed generator τ of $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$. In the notation of Lemma 5.3, we have $\varphi_u^x = \varphi_u^b = \tau\varphi_u$ for some $x \in P_d$ satisfying $|\det(x)| = (q+1)_2$, by Corollary 4.3 and Lemmas 5.1 and 5.2. Then by Lemma 5.3, there is a conjugate χ_0 of χ such that $\chi_0^x = \tau\chi_0$. In particular, note that the condition on the determinant yields that $\chi_0^x \neq \chi_0$, so $\tau\chi_0 \neq \chi_0$. Since χ and χ_0 have the same field of values, we see $\tau\chi \neq \chi$, and we have $\mathbb{F}_\lambda \neq \mathbb{Q}(\chi)$. □

Proposition 6.6. *Let* $\epsilon = -1$ *and suppose that* $q \equiv 3 \bmod 4$ *and* $4 \leq n_2 \leq (q+1)_2$*, and let* $\chi \in \mathrm{Irr}(G|\widetilde{\chi}_\lambda)$ *and* $I := \mathrm{stab}_{\widetilde{G}}(\chi)$*. Then* $\mathbb{F}_\lambda = \mathbb{Q}(\chi)$ *if and only if* $[\widetilde{G} : I]_2 < n_2$*.*

Proof. First, note that $[\widetilde{G} : I]_2 \leq n_2$ by Proposition 2.5. Note that by Lemma 3.5, we may assume that $[\widetilde{G} : I]$ is even and therefore that $\chi \in \mathrm{Irr}(G|\widetilde{\Gamma}_u)$ where u has no odd-power elementary divisor, by Proposition 6.4. By Lemmas 4.5, 5.1, and 5.2, there is some $x \in \widetilde{G}$ such that $\varphi_u^x = \varphi_u^b = \tau\varphi_u$ and $|\det(x)| = 2(q+1)_2/n_2$, which is divisible by 2 since $n_2 \leq (q+1)_2$. By Lemma 5.3, there is a conjugate χ_0 of χ such that $\chi_0^x = \tau\chi_0$.

Suppose first that $[\widetilde{G} : I]_2 = n_2$, so that x cannot stabilize χ_0, since $[I : G]_2 = (q+1)_2/n_2$ (and the same is true for the stabilizer of χ_0). This yields that $\chi_0 \neq \tau\chi_0$, so the same holds for χ. Hence if $[\widetilde{G} : I]_2 = n_2$, then $\mathbb{F}_\lambda \neq \mathbb{Q}(\chi)$.

Now suppose $[\widetilde{G} : I]_2 < n_2$. That is, $[\widetilde{G} : I]_2 \leq n_2/2$. Then the stabilizer of χ_0 must contain x, since $I/G \cong \mathcal{I}(\lambda)$ is cyclic and contains the unique subgroup of $\widetilde{G}/G$ of size $2(q+1)_2/n_2$. Then $\chi_0 = \tau\chi_0$, and the same is true for χ, so $\mathbb{F}_\lambda = \mathbb{Q}(\chi)$. □

Proof of Theorem 6.1. For $\epsilon = 1$, this is [Turull 2001, Theorem 4.8], so we assume $\epsilon = -1$. When q is odd, note the case $[\widetilde{G} : I]_2 = n_2$, for any n even, is equivalent to having $\alpha\lambda = \lambda$ for any $\alpha \in \widehat{T_1}$ of order n_2. In the case $n \equiv 2 \bmod 4$, we remark that this α is sgn. Hence Propositions 3.3, 6.5, and 6.6 combine to yield the statement. □

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