GALOIS AUTOMORPHISMS AND A UNIQUE JORDAN DECOMPOSITION IN THE CASE OF CONNECTED CENTRALIZER

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ABSTRACT. We show that the Jordan decomposition of characters of finite reductive groups can be chosen so that if the centralizer of the relevant semisimple element in the dual group is connected, then the map is Galois-equivariant. Further, in this situation, we show that there is a unique Jordan decomposition satisfying conditions analogous to those of Digne–Michel's unique Jordan decomposition in the connected center case.

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1. INTRODUCTION

Given a finite group G, the fields of values of the irreducible complex characters of G, $\mathrm{Irr}(G)$, have revealed themselves to be valuable and interesting number-theoretic data corresponding to the structure of G. Numerous examples of results demonstrating this involve the rational-valued characters of G, real-valued characters of G, and the question of whether a representation of G can be realized over the field of values of its characters. Thus, the Galois action on these fields of character values becomes a key problem in the character theory of finite groups.

In this paper, we study the action of $\mathcal{G} := \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ on the set $\mathrm{Irr}(G)$, where G is a finite reductive group. This family of groups is of particular interest because of their role as algebraic groups, their actions on finite geometries, and their relation to finite simple groups. In particular, for finite groups of Lie type, a key to understanding the action of $\mathcal{G}$ on the set $\mathrm{Irr}(G)$ is understanding how various parametrizations of the set $\mathrm{Irr}(G)$ behave under the action of $\mathcal{G}$. In [26, 27], Srinivasan and the third-named author study this question for the Jordan decomposition of characters, in the case that the underlying algebraic group has a connected center. In [22], the first-named author studies this question for the Howlett–Lehrer parametrization of Harish-Chandra series. More results have been obtained in [8] for the case of connected center, and the authors have studied the question of fields of values of characters in [25, 24].

The question of the action of $\mathcal{G}$ on $\mathrm{Irr}(G)$ is particularly difficult in the case that the underlying algebraic group has disconnected center, and will play a crucial role in, for example, proving the inductive Galois–McKay conditions of [18] to prove the McKay–Navarro conjecture for odd primes. (Note that for the prime 2, this was finished in [20, 21].)

The results of [26, 27] make essential use of the unique Jordan decomposition proved by Digne and Michel in [6] in the case of connected center. The goal of the present paper is twofold: we extend the results of [27] on the action of $\mathcal{G}$ on Jordan decomposition to the case that the underlying group does not necessarily have a connected center, but that the semisimple element in question has a connected centralizer in the dual group; and we extend the results of [6] to show that there is a unique Jordan decomposition satisfying certain properties in the same situation. In this context, we may embed the underlying connected reductive group $\mathbf{G}$ into another connected reductive group

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$\widetilde{\mathbf{G}}$ with a connected center, using a so-called regular embedding $\iota\colon \mathbf{G} \hookrightarrow \widetilde{\mathbf{G}}$, which in turn yields a dual surjection $\iota^*\colon \widetilde{\mathbf{G}}^* \to \mathbf{G}^*$ of the dual groups. Our main result is the following:

Theorem 1.1. *Let $\mathbf{G}$ be a connected reductive group and $F\colon \mathbf{G} \to \mathbf{G}$ a Frobenius endomorphism, and let $G = \mathbf{G}^F$ be the corresponding group of Lie type. Let $(\mathbf{G}^*, F^*)$ be dual to $(\mathbf{G}, F)$ and $s \in G^* := (\mathbf{G}^*)^{F^*}$ a semisimple element such that $(\mathrm{C}_{\mathbf{G}^*}(s)/\mathrm{C}^\circ_{\mathbf{G}^*}(s))^{F^*} = 1$, or equivalently $\mathrm{C}_{\mathbf{G}^*}(s)^{F^*} \leqslant \mathrm{C}^\circ_{\mathbf{G}^*}(s)$.*

(1) *Given any regular embedding $\iota\colon \mathbf{G} \to \widetilde{\mathbf{G}}$, there is a Jordan decomposition map $J_{s,\iota}\colon \mathcal{E}(G, s) \to \mathcal{E}(\mathrm{C}_{G^*}(s), 1)$ that is invariant under the choice of preimage $\widetilde{s} \in (\widetilde{\mathbf{G}}^*)^{F^*}$ such that $\iota^*(\widetilde{s}) = s$, and such that the collection $\{J_{s,\iota} \mid \mathrm{C}_{\mathbf{G}^*}(s)^{F^*} \leqslant \mathrm{C}^\circ_{\mathbf{G}^*}(s)\}$ is $\mathcal{G}$-equivariant (in the sense of Lemma 4.2 below).*
(2) *Further, there exists a unique Jordan decomposition map $J_s\colon \mathcal{E}(G, s) \to \mathcal{E}(\mathrm{C}_{G^*}(s), 1)$ satisfying properties (1)-(7) of Condition 5.1 below. (In particular, the collection $\{J_s \mid \mathrm{C}_{\mathbf{G}^*}(s)^{F^*} \leqslant \mathrm{C}^\circ_{\mathbf{G}^*}(s)\}$ is also $\mathcal{G}$-equivariant.)*

1.1. **Notation.** For any group G and element $g \in G$ we denote by $\mathrm{Ad}_g : G \to G$ the inner automorphism defined by $\mathrm{Ad}_g(x) = gxg^{-1}$. If $H \leqslant G$ is a subgroup, then we obtain an isomorphism $\mathrm{Ad}_g : H \to {}^gH = gHg^{-1}$, which we also denote by Ad_g.

Suppose now that G is finite. We write $\mathrm{cf}(G)$ for the space of complex-valued class functions on G and $\mathrm{Irr}(G) \subseteq \mathrm{cf}(G)$ for the set of complex irreducible characters of G. We let $\mathbb{1} \in \mathrm{Irr}(G)$ denote the trivial character of G. If $H \leqslant G$ is a subgroup, then for any $\varphi \in \mathrm{cf}(H)$ and $\chi \in \mathrm{cf}(G)$ we use $\mathrm{Ind}_H^G(\varphi)$ and $\mathrm{Res}_H^G(\chi)$ to denote induction from H to G and restriction from G to H, respectively. Further, we let $\mathrm{Irr}(G|\varphi)$ denote the irreducible constituents of $\mathrm{Ind}_H^G(\varphi)$ and $\mathrm{Irr}(H|\chi)$ denote the irreducible constituents of $\mathrm{Res}_H^G(\chi)$. In particular, $\mathrm{Irr}(G|\chi)$ is the set of irreducible constituents of $\chi \in \mathrm{cf}(G)$.

If $\phi\colon G \to H$ is a homomorphism between finite groups, we write ${}^\top\phi$ for the function ${}^\top\phi\colon \mathrm{cf}(H) \to \mathrm{cf}(G)$ defined by ${}^\top\phi(\chi) = \chi \circ \phi$. In particular, if ϕ is injective and we identify G with $\phi(G)$, then this map can be viewed as restriction; similarly if ϕ is surjective then this map can be viewed as inflation through the quotient map $G \to G/\ker\phi$.

2. Digne–Michel's Unique Jordan Decomposition

In this section, we develop some basic notation and recall the main result of [6]. Throughout, $p > 0$ will be a fixed prime integer and $\mathbb{F} = \overline{\mathbb{F}}_p$ will be an algebraic closure of the finite field of cardinality p. All algebraic groups are assumed to be affine $\mathbb{F}$-varieties.

If $\mathbf{T}$ is a torus, then we denote by $X(\mathbf{T})$ and $\check{X}(\mathbf{T})$ the character and cocharacter groups of $\mathbf{T}$. Recall that for any two tori $\mathbf{T}$ and $\mathbf{T}'$ we have bijections

$$\mathrm{Hom}(\mathbf{T}, \mathbf{T}') \xrightarrow{X(-)} \mathrm{Hom}_{\mathbb{Z}}(X(\mathbf{T}'), X(\mathbf{T})) \xrightarrow{\widetilde{(-)}} \mathrm{Hom}_{\mathbb{Z}}(\check{X}(\mathbf{T}), \check{X}(\mathbf{T}')) \xleftarrow{\check{X}(-)} \mathrm{Hom}(\mathbf{T}, \mathbf{T}')$$

where $\mathrm{Hom}(\mathbf{T}, \mathbf{T}')$ denotes the set of homomorphisms of algebraic groups. If $\phi \in \mathrm{Hom}(\mathbf{T}, \mathbf{T}')$ then $X(\phi)$ is the map $\chi \mapsto \chi \circ \phi$ and $\check{X}(\phi)$ is the map $\gamma \mapsto \phi \circ \gamma$.

If $\mathbf{G}$ is connected reductive and $\mathbf{T} \leqslant \mathbf{G}$ is a maximal torus, then we denote by $\Phi_{\mathbf{G}}(\mathbf{T}) \subseteq X(\mathbf{T})$ and $\check{\Phi}_{\mathbf{G}}(\mathbf{T}) \subseteq \check{X}(\mathbf{T})$ the roots and coroots of $\mathbf{G}$, respectively. We also call $W = \mathrm{N}_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ the Weyl group of $\mathbf{G}$ (with respect to $\mathbf{T}$).

Now let $(\mathbf{G}, F)$ be a pair consisting of an algebraic group $\mathbf{G}$ and a Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$. Following Steinberg [28, p. 78] we say $(\mathbf{G}, F)$ is F-simple if $\mathbf{G} = \mathbf{G}_1 \cdots \mathbf{G}_n$ is an almost direct product of quasisimple groups permuted cyclically by F. We refer to the type of $(\mathbf{G}, F)$ as the type of the underlying root system of $\mathbf{G}_1$.

If $\mathbf{G}$ is connected reductive, then we refer to the pair $(\mathbf{G}, F)$, or the finite group of fixed points $G = \mathbf{G}^F \leqslant \mathbf{G}$, as a finite reductive group. The morphism $\mathscr{L} = \mathscr{L}_F = \mathscr{L}_{\mathbf{G},F} : \mathbf{G} \to \mathbf{G}$, defined by $\mathscr{L}(g) = g^{-1}F(g)$, is called the Lang map of $(\mathbf{G}, F)$. It is surjective when $\mathbf{G}$ is connected.

Let $(\mathbf{G}, F)$ and $(\mathbf{G}^*, F^*)$ be two finite reductive groups and assume $(\mathbf{T}, \mathbf{T}^*, \delta)$ is a triple consisting of: an F-stable maximal torus $\mathbf{T} \leqslant \mathbf{G}$, an F^*-stable maximal torus $\mathbf{T}^* \leqslant \mathbf{G}^*$, and an isomorphism $\delta : X(\mathbf{T}) \to \check{X}(\mathbf{T}^*)$ satisfying

$$\check{X}(F^*) \circ \delta = \delta \circ X(F). \tag{2.1}$$

We say $(\mathbf{G}, F)$ and $(\mathbf{G}^*, F^*)$ are *dual* if for some triple $\mathscr{T} = (\mathbf{T}, \mathbf{T}^*, \delta)$ we have δ is an isomorphism of root data, see [7, Def. 11.1.10]. We call $\mathscr{T}$ a *witness* to the duality and $\mathcal{D} = ((\mathbf{G}, F), (\mathbf{G}^*, F^*), \mathscr{T})$ a *rational duality*. Note that this induces a duality between the corresponding Weyl groups $W = \mathrm{N}_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ and $W^* = \mathrm{N}_{\mathbf{G}^*}(\mathbf{T}^*)/\mathbf{T}^*$.

Let us fix once and for all an embedding $\mathbb{F}^\times \hookrightarrow \mathbb{C}^\times$ and an isomorphism $\mathbb{F}^\times \to (\mathbb{Q}/\mathbb{Z})_{p'}$. If two tori $(\mathbf{T}, F)$ and $(\mathbf{T}^*, F^*)$ are dual, then each isomorphism $\delta : X(\mathbf{T}) \to \check{X}(\mathbf{T}^*)$ satisfying (2.1) determines a group isomorphism $\mathbf{T}^{*F^*} \to \mathrm{Irr}(\mathbf{T}^F)$ which we denote by $s \mapsto \hat{s}$, see [7, Prop. 11.1.14]. This depends on δ and our preceding choices of embedding $\mathbb{F}^\times \hookrightarrow \mathbb{C}^\times$ and isomorphism $\mathbb{F}^\times \to (\mathbb{Q}/\mathbb{Z})_{p'}$.

Assume we have a rational duality and let $G^* := \mathbf{G}^{*F^*}$ be the finite dual group. Given a semisimple element $s \in G^*$, we denote by $\mathcal{E}(G, s) \subseteq \mathrm{Irr}(G)$ the *rational Lusztig series* corresponding to the G^*-conjugacy class of s, see [7, Def. 12.4.3]. Lusztig [15] has shown that there exists a (not necessarily canonical) Jordan decomposition map $\mathcal{E}(G, s) \to \mathcal{E}(\mathrm{C}_{G^*}(s), 1)$. Assuming $\mathrm{Z}(\mathbf{G})$ is connected, Digne and Michel [6] have shown that this map can be chosen uniquely to satisfy certain nice properties.

Theorem 2.1 (Digne–Michel, 1990)**.** *For each rational duality $\mathcal{D} = ((\mathbf{G}, F), (\mathbf{G}^*, F^*), \mathscr{T}_0)$ with $\mathrm{Z}(\mathbf{G})$ connected and each semisimple element $s \in \mathbf{G}^{*F^*}$, there exists a unique bijection*

$$J_s^{\mathbf{G}} : \mathcal{E}(G, s) \longrightarrow \mathcal{E}(\mathrm{C}_{G^*}(s), 1)$$

such that the following properties hold:

1. *If $\mathbf{T} \leqslant \mathbf{G}$ and $\mathbf{T}^* \leqslant \mathbf{G}^*$ are dual maximal tori with $\mathbf{T}^* \leqslant \mathrm{C}_{\mathbf{G}^*}(s)$ then for any $\chi \in \mathcal{E}(G, s)$ we have*
$$\langle \chi, R_{\mathbf{T}}^{\mathbf{G}}(\widehat{s}) \rangle = \langle J_s^{\mathbf{G}}(\chi), \epsilon_{\mathbf{G}} \epsilon_{\mathrm{C}_{\mathbf{G}^*}(s)} R_{\mathbf{T}^*}^{\mathrm{C}_{\mathbf{G}^*}(s)}(\mathbb{1}) \rangle.$$
2. *If $s = 1$ then:*
   - (a) *Let d be the smallest positive integer such that F^d is a split Frobenius endomorphism, see* [7, Def. 4.3.2]. *The eigenvalues of F^d associated to χ are equal, up to an integer power of $q^{d/2}$, to the eigenvalues of F^{*d} associated to $J_1^{\mathbf{G}}(\chi)$.*
   - (b) *If χ is in the principal series then $J_1^{\mathbf{G}}(\chi)$ and χ correspond to the same character of the Hecke algebra $\mathrm{End}_{\mathbb{C}G}(\mathrm{Ind}_B^G(\mathbb{1}))$, where $B = \mathbf{B}^F$ and $\mathbf{B}$ is an F-stable Borel subgroup of $\mathbf{G}$.*
3. *If $z \in \mathrm{Z}(G^*)$ is central and $\chi \in \mathcal{E}(G, s)$, then $J_{sz}^{\mathbf{G}}(\chi \otimes \hat{z}) = J_s^{\mathbf{G}}(\chi)$.*
4. *If $\mathbf{L} \leqslant \mathbf{G}$ and $\mathbf{L}^* \leqslant \mathbf{G}^*$ are dual Levi subgroups with $\mathrm{C}_{\mathbf{G}^*}(s) \leqslant \mathbf{L}^*$ then the following diagram is commutative:*

$$\begin{array}{ccc} \mathbb{Z}\mathcal{E}(G, s) & \xrightarrow{J_s^{\mathbf{G}}} & \mathbb{Z}\mathcal{E}(\mathrm{C}_{G^*}(s), 1) \\ \uparrow{\scriptstyle R_{\mathbf{L}}^{\mathbf{G}}} & & \| \\ \mathbb{Z}\mathcal{E}(L, s) & \xrightarrow{J_s^{\mathbf{L}}} & \mathbb{Z}\mathcal{E}(\mathrm{C}_{L^*}(s), 1) \end{array}$$

Here $R_{\mathbf{L}}^{\mathbf{G}}$ denotes Lusztig's twisted induction and we extend $J_s^{\mathbf{G}}$ by linearity to a map on class functions.

(5) *Assume* $(\mathbf{G}, F)$ *is* F*-simple of type* E_8 *and* $(\mathrm{C}_{\mathbf{G}^*}(s), F^*)$ *is of type* $E_7.A_1$ *(respectively,* $E_6.A_2$*, respectively* ${}^2E_6.{}^2A_2$*). If* $\mathbf{L} \leqslant \mathbf{G}$ *and* $\mathbf{L}^* \leqslant \mathbf{G}^*$ *are dual Levi subgroups of type* E_7 *(respectively,* E_6*, respectively* E_6*) with* $s \in \mathrm{Z}(\mathbf{L}^*)$ *then the following diagram is commutative:*

$$\begin{array}{ccc} \mathbb{Z}\mathcal{E}(G,s) & \xrightarrow{J_s^{\mathbf{G}}} & \mathbb{Z}\mathcal{E}(\mathrm{C}_{G^*}(s),1) \\ \uparrow R_{\mathbf{L}}^{\mathbf{G}} & & \uparrow R_{\mathbf{L}^*}^{\mathrm{C}_{\mathbf{G}^*}(s)} \\ \mathbb{Z}\mathcal{E}(L,s)^{\bullet} & \xrightarrow{J_s^{\mathbf{L}}} & \mathbb{Z}\mathcal{E}(L^*,1)^{\bullet} \end{array}$$

where the superscript $\bullet$ *denotes the cuspidal part of the Lusztig series.*

(6) *Given an epimorphism* $\varphi : (\mathbf{G}, F) \to (\mathbf{G}_1, F_1)$*, with kernel* $\ker(\varphi)$ *a central torus, the following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{E}(G,s) & \xrightarrow{J_s^{\mathbf{G}}} & \mathcal{E}(\mathrm{C}_{G^*}(s),1) \\ \uparrow {}^{\top}\varphi & & \downarrow {}^{\top}\varphi^* \\ \mathcal{E}(G_1,s_1) & \xrightarrow{J_{s_1}^{\mathbf{G}_1}} & \mathcal{E}(\mathrm{C}_{G_1^*}(s_1),1), \end{array}$$

where $s_1 \in G_1^*$ *is the unique semisimple element such that* $s = \varphi^*(s_1) \in G^*$*.*

(7) *If* $\mathbf{G} = \prod_i \mathbf{G}_i$ *is a direct product of* F*-stable subgroups, then* $J^{\mathbf{G}}_{\prod_i s_i} = \prod_i J^{\mathbf{G}_i}_{s_i}$*.*

We close by making a few remarks. Let $\mathscr{T} = (\mathbf{T}_0, \mathbf{T}_0^\star, \delta_0)$ be the witness to the duality occurring in the statement of Theorem 2.1. In (1) of the theorem, we must choose an isomorphism $\delta : X(\mathbf{T}) \to \check{X}(\mathbf{T}^\star)$ for the map $s \mapsto \hat{s}$ to be defined. In other words, such a choice is needed to make $(\mathbf{T}, F)$ and $(\mathbf{T}^*, F^*)$ dual. Moreover, in (4) and (5) of the statement, we must choose a witness to the duality of $(\mathbf{L}, F)$ and $(\mathbf{L}^*, F^*)$ for the bijection $J_s^{\mathbf{L}}$ to be defined. We briefly recall how this is done.

For any $(g, g^*) \in \mathbf{G} \times \mathbf{G}^*$ we may consider the tuple

$$(g, g^*) \cdot \mathscr{T}_0 = ({}^g\mathbf{T}, {}^{g^*}\mathbf{T}_0^\star, \check{X}(\mathrm{Ad}_{g^*}) \circ \delta_0 \circ X(\mathrm{Ad}_g)).$$

In general, this will not be a witness to the duality between $(\mathbf{G}, F)$ and $(\mathbf{G}^*, F^*)$. Precisely when this is the case is described in [5, Lem. 4.3.3].

To say that two Levi subgroups $\mathbf{L} \leqslant \mathbf{G}$ and $\mathbf{L}^* \leqslant \mathbf{G}^*$ are dual is to say that some $\mathscr{T} = (g, g^*) \cdot \mathscr{T}_0$ is a witness to the duality between $(\mathbf{L}, F)$ and $(\mathbf{L}^*, F^*)$. If $\mathscr{T} = (\mathbf{T}, \mathbf{T}^*, \delta)$ then this is equivalent to requiring that: $\mathscr{T}$ is a witness to the duality between $(\mathbf{G}, F)$ and $(\mathbf{G}^*, F^*)$, $\mathbf{T} \leqslant \mathbf{L}$, $\mathbf{T}^* \leqslant \mathbf{L}^*$, and $\delta(\Phi_{\mathbf{L}}(\mathbf{T})) = \check{\Phi}_{\mathbf{L}^*}(\mathbf{T}^*)$.

Finally, let us point out that condition (2a) of Theorem 2.1 has been strengthened by Srinivasan–Vinroot in [27] to the following:

(2a*) If $s = 1$ then the eigenvalues of F^d associated to χ are equal, up to an integer power of q^d, to the eigenvalues of F^{*d} associated to $J_1^{\mathbf{G}}(\chi)$.

3. Clifford Theory in the Connected Centralizer Case

Given a semisimple element $s \in \mathbf{G}^*$, we write $A_{\mathbf{G}}(s) := \mathrm{C}_{\mathbf{G}^*}(s)/\mathrm{C}^\circ_{\mathbf{G}^*}(s)$. Here $\mathrm{C}^\circ_{\mathbf{G}^*}(s) := \mathrm{C}_{\mathbf{G}^*}(s)^\circ$. (When there is no confusion, we simply write $A(s) := A_{\mathbf{G}}(s)$.) In the present paper, we adjust Digne–Michel's unique Jordan decomposition in the case $\mathrm{Z}(\mathbf{G})$ is connected (Theorem 2.1) to allow for the case that $\mathrm{Z}(\mathbf{G})$ is disconnected but $A(s)^F = 1$. That is, the case $\mathrm{C}_{\mathbf{G}^*}(s)^F = \mathrm{C}^\circ_{\mathbf{G}^*}(s)^F$. To do this, we use the concept of a regular embedding, as in [15, Section 7].

As in [15], we define a *regular embedding* to be an injective homomorphism $\iota \colon \mathbf{G} \hookrightarrow \widetilde{\mathbf{G}}$, where $(\widetilde{\mathbf{G}}, \widetilde{F})$ is a connected reductive group defined over $\mathbb{F}_q$ such that $\mathrm{Z}(\widetilde{\mathbf{G}})$ is connected, ι commutes with the Frobenius morphisms in the sense that $\iota \circ F = \widetilde{F} \circ \iota$, the map ι induces an isomorphism of $\mathbf{G}$ with a closed subgroup $\iota(\mathbf{G})$ of $\widetilde{\mathbf{G}}$, and $[\iota(\mathbf{G}), \iota(\mathbf{G})] = [\widetilde{\mathbf{G}}, \widetilde{\mathbf{G}}]$. Given such a map, there is a corresponding dual surjection $\iota^*\colon \widetilde{\mathbf{G}}^* \to \mathbf{G}^*$, such that $\ker \iota^*$ is a central torus, see Definition 7.3 for further details.

Given a fixed regular embedding ι, we will write $G := \mathbf{G}^F$, $\widetilde{G} := \widetilde{\mathbf{G}}^{\widetilde{F}}$, $G^* := \mathbf{G}^{*F^*}$, and $\widetilde{G}^* := (\widetilde{\mathbf{G}}^*)^{\widetilde{F}^*}$. (For the time being, we do not specify the regular embedding, but in Section 8, we will fix a specific regular embedding $\bar{\iota}$ to prove Theorem 1.1.) By an abuse of notation, we will sometimes write simply F for $\widetilde{F}$, F^*, and $\widetilde{F}^*$. We will also identify G with $\iota(\mathbf{G})^F$, so we view G as a subgroup of $\widetilde{G}$. Recall that Theorem 2.1 yields a unique Jordan decomposition for $\widetilde{G}$.

By [9, Prop. 2.6.16], if $s \in G^*$ is semisimple, then $\mathcal{E}(G,s)$ is exactly the set of irreducible constituents of the restrictions of characters from $\mathcal{E}(\widetilde{G},\widetilde{s})$, where $\widetilde{s} \in \widetilde{G}^*$ is any semisimple element satisfying $\iota^*(\widetilde{s}) = s$.

Now, recall that $\widetilde{G}/G$ is abelian and has multiplicity-free restrictions, by [15, Prop. 10] (see also Theorem 6.7). Let $\chi \in \mathcal{E}(G,s)$ and let $\widetilde{G}_\chi$ denote the stabilizer in $\widetilde{G}$ of χ. Then χ extends to $\widetilde{G}_\chi$ and by Clifford theory, the characters of $\widetilde{G}$ above χ are of the form $\mathrm{Ind}_{\widetilde{G}_\chi}^{\widetilde{G}}(\bar{\chi}\beta)$ where $\bar{\chi}$ is a fixed extension of χ to $\widetilde{G}_\chi$ and β ranges through characters of $\widetilde{G}_\chi/G$. Note that $\beta \in \mathrm{Irr}(\widetilde{G}_\chi/G)$ can be viewed as $\mathrm{Res}_{\widetilde{G}_\chi}^{\widetilde{G}}(\widetilde{\beta})$ for some $\widetilde{\beta} \in \mathrm{Irr}(\widetilde{G}/G)$. Then $\mathrm{Ind}_{\widetilde{G}_\chi}^{\widetilde{G}}(\bar{\chi}\beta) = \mathrm{Ind}_{\widetilde{G}_\chi}^{\widetilde{G}}(\bar{\chi})\widetilde{\beta}$ by [10, Prob. (5.3)]. Now by [6, Prop. 2.6 and 2.7], $\widetilde{\beta}$ is of the form $\widehat{z}$ for some $z \in \mathrm{Z}(\widetilde{G}^*)$ and $\mathrm{Ind}_{\widetilde{G}_\chi}^{\widetilde{G}}(\bar{\chi})\widetilde{\beta} \in \mathcal{E}(\widetilde{G},\widetilde{s}z)$ where $\mathrm{Ind}_{\widetilde{G}_\chi}^{\widetilde{G}}(\bar{\chi}) \in \mathcal{E}(\widetilde{G},\widetilde{s})$.

By [6, Prop. 2.5], $\mathcal{E}(\widetilde{G},\widetilde{s})$ and $\mathcal{E}(\widetilde{G},\widetilde{s}z)$ are equal if and only if $z \in [[A,s]]$, where $[[A,s]] \cong A(s)$ is the set of commutators of preimages in $\widetilde{\mathbf{G}}^*$ of $a \in A(s)$ and s. In our case, since $z \in \widetilde{G}^*$, we have $z \in [[A,s]]^F \cong A(s)^F$. In particular, we have

Lemma 3.1. *Keep the notation above. If $|A_{\mathbf{G}}(s)^F| = 1$, then every character of $\widetilde{G}$ above χ lies in a different Lusztig series.*

Now, given $\widetilde{\chi} \in \mathcal{E}(\widetilde{G},\widetilde{s})$, the number of constituents of $\mathrm{Res}_G^{\widetilde{G}}(\widetilde{\chi})$ is the number of $\widetilde{\beta} = \hat{z} \in \mathrm{Irr}(\widetilde{G}/G)$ such that $\widetilde{\chi}\widetilde{\beta} = \widetilde{\chi}$, see [7, Lem. 11.3.9]. But if $\widetilde{\chi}\hat{z} = \widetilde{\chi}$, then the corresponding element $z \in \mathrm{Z}(\widetilde{G}^*)$ satisfies $\widetilde{s}z$ is $\widetilde{G}^*$-conjugate to $\widetilde{s}$. But the number of such z is $|A(s)^F|$, as above (see also [3, Cor. 2.8]).

Hence, if $|A(s)^F| = 1$, then further every $\chi \in \mathcal{E}(G,s)$ extends to $\widetilde{G}$. That is:

Lemma 3.2. *Keep the notation above. Let $s \in G^*$ be semisimple such that $|A_{\mathbf{G}}(s)^F| = 1$. Then for any $\widetilde{s} \in \widetilde{G}^*$ with $\iota^*(\widetilde{s}) = s$, the restriction map induces a bijection $\mathrm{Res}_G^{\widetilde{G}}\colon \mathcal{E}(\widetilde{G},\widetilde{s}) \to \mathcal{E}(G,s)$.*

4. Defining the Jordan Decomposition

Keep the notation from Section 3. We aim to show that the Jordan decomposition for G may also be chosen in a nice way when $A(s)^F = 1$. The next lemma establishes the map we will work with.

Lemma 4.1. *Let $\iota\colon \mathbf{G} \to \widetilde{\mathbf{G}}$ be a regular embedding, let $s \in G^*$ be semisimple, and let $\widetilde{s} \in \widetilde{G}^*$ such that $\iota^*(\widetilde{s}) = s$. Let $J_{\widetilde{s}}$ be the unique Jordan decomposition $\mathcal{E}(\widetilde{G},\widetilde{s}) \to \mathcal{E}(\mathrm{C}_{\widetilde{G}^*}(\widetilde{s}),1)$ guaranteed by Digne and Michel (see Theorem 2.1). Assume $A_{\mathbf{G}}(s)^F = 1$. Then the unique map $J_{s,\iota}$ for which the following diagram commutes:*

$$\begin{array}{ccc}
\mathcal{E}(\widetilde{G},\widetilde{s}) & \xrightarrow{J_{\widetilde{s}}} & \mathcal{E}(\mathrm{C}_{\widetilde{G}^*}(\widetilde{s}),1) \\
\Big\downarrow \mathrm{Res}_G^{\widetilde{G}} & & \Big\uparrow {}^{\top}\iota^* \\
\mathcal{E}(G,s) & \xrightarrow{J_{s,\iota}} & \mathcal{E}(\mathrm{C}_{G^*}(s),1)
\end{array}$$

is independent of the choice of $\widetilde{s}$.

Proof. Let $\widetilde{s}$ and $\widetilde{s}'$ be two preimages of s in $\widetilde{G}^*$ under ι^*. Then $\widetilde{s}'$ must be of the form $\widetilde{s}z$ for $z \in \mathrm{Z}(\widetilde{G}^*)$, so the characters of $\mathcal{E}(\widetilde{G}, \widetilde{s}')$ are just $\widetilde{\chi} \otimes \widehat{z}$ for $\widetilde{\chi} \in \mathcal{E}(\widetilde{G}, \widetilde{s})$. Hence it suffices to note that $\mathrm{Res}^{\widetilde{G}}_G(\widetilde{\chi} \otimes \widehat{z}) = \mathrm{Res}^{\widetilde{G}}_G(\widetilde{\chi})$ and $\mathrm{C}_{\widetilde{G}^*}(\widetilde{s}) = \mathrm{C}_{\widetilde{G}^*}(\widetilde{s}z)$ and apply property (3) of Theorem 2.1. □

Our next goal is to show that, given a fixed regular embedding ι, the map $J_{s,\iota}$ in Lemma 4.1 above yields a $\mathcal{G}$-equivariant Jordan decomposition map. The next lemma shows that the collection of maps $\{J_{s,\iota} \mid A(s)^F = 1\}$ is $\mathcal{G}$-equivariant. By the proof of [23, Lemma 3.4], there is some $s^\sigma \in G^*$ such that $\mathcal{E}(G, s)^\sigma = \mathcal{E}(G, s^\sigma)$. (Namely, if σ maps $|s|$th roots of unity to the kth power with $(|s|, k) = 1$, then we have $s^\sigma = s^k$.) Further, if $A(s)^F = 1$, then $A(s^\sigma)^F = 1$ as well.

The main result of [27] is that when $\mathrm{Z}(\mathbf{G})$ is connected and J_s is the map as in Theorem 2.1, we have $J_{s^\sigma}(\chi^\sigma) = J_s(\chi)^\sigma$. We prove the analogous statement for the map defined in Lemma 4.1.

Lemma 4.2. *Let $s \in G^*$ be semisimple with $A(s)^F = 1$, let $\iota \colon \mathbf{G} \to \widetilde{\mathbf{G}}$ be a regular embedding, and let $J_{s,\iota}$ be the map in Lemma 4.1. Then for any $\chi \in \mathcal{E}(G, s)$ and $\sigma \in \mathcal{G}$, we have $J_{s^\sigma,\iota}(\chi^\sigma) = J_{s,\iota}(\chi)^\sigma$.*

Proof. Let $\chi \in \mathcal{E}(G, s)$ and $\sigma \in \mathcal{G}$. Letting $\widetilde{s} \in \widetilde{G}^*$ be such that $\iota^*(\widetilde{s}) = s$, we have by Lemma 3.2 and the discussion preceding it that there is a unique $\widetilde{\chi} \in \mathcal{E}(\widetilde{G}, \widetilde{s})$ lying above χ and a unique character in $\mathcal{E}(\widetilde{G}, \widetilde{s})^\sigma = \mathcal{E}(\widetilde{G}, \widetilde{s}^\sigma)$ above χ^σ. By uniqueness, the latter must therefore be equal to $\widetilde{\chi}^\sigma$. Hence the left-hand map $\mathrm{Res}^{\widetilde{G}}_G$ is $\mathcal{G}$-equivariant. The right-hand map is inflation, which is also $\mathcal{G}$-equivariant. Finally, the main result of [27] gives $J_{\widetilde{s}}(\widetilde{\chi})^\sigma = J_{\widetilde{s}^\sigma}(\widetilde{\chi}^\sigma)$, which forces the claim. □

Next, we show that $J_{s,\iota}$ is indeed a Jordan decomposition.

Lemma 4.3. *Let $s \in G^*$ be semisimple with $A(s)^F = 1$, let ι be a regular embedding, and let $\mathbf{T} \leqslant \mathbf{G}$ and $\mathbf{T}^* \leqslant \mathbf{G}^*$ be dual maximal tori with $\mathbf{T}^* \leqslant \mathrm{C}_{\mathbf{G}^*}(s)$. If $J_{s,\iota}$ is the map in Lemma 4.1, then $J_{s,\iota}$ is a Jordan decomposition, in the sense that*

$$\langle \chi, R^{\mathbf{G}}_{\mathbf{T}}(\widehat{s})\rangle_G = \langle J_{s,\iota}(\chi), \epsilon_{\mathbf{G}}\epsilon_{\mathrm{C}^\circ_{\mathbf{G}^*}(s)} R^{\mathrm{C}_{\mathbf{G}^*}(s)}_{\mathbf{T}^*}(\mathbb{1})\rangle_{\mathrm{C}_{G^*}(s)}$$

for any $\chi \in \mathcal{E}(G, s)$.

Proof. This statement follows from the more general case covered in [15], also appearing in [4, 15.2], but we give some details here for the sake of clarity. Let $\widetilde{\chi} \in \mathcal{E}(\widetilde{G}, \widetilde{s})$ and $\chi = \mathrm{Res}^{\widetilde{G}}_G(\widetilde{\chi})$. We have

$$\langle \chi, R^{\mathbf{G}}_{\mathbf{T}}(\widehat{s})\rangle_G = \langle \mathrm{Res}^{\widetilde{G}}_G(\widetilde{\chi}), R^{\mathbf{G}}_{\mathbf{T}}(\widehat{s})\rangle_G = \langle \widetilde{\chi}, \mathrm{Ind}^{\widetilde{G}}_G(R^{\mathbf{G}}_{\mathbf{T}}(\widehat{s}))\rangle_{\widetilde{G}},$$

by Frobenius reciprocity. We have $\mathrm{Ind}^{\widetilde{G}}_G(R^{\mathbf{G}}_{\mathbf{T}}(\widehat{s})) = R^{\widetilde{\mathbf{G}}}_{\widetilde{\mathbf{T}}}(\mathrm{Ind}^{\widetilde{T}}_T(\widehat{s}))$, by [2, Cor. 2.1.3]. Then $\mathrm{Ind}^{\widetilde{T}}_T(\widehat{s})$ is a multiplicity-free sum of linear characters of $\widetilde{T}$, exactly one of which is $\widehat{\widetilde{s}}$. By disjointness of Lusztig series, we then have

$$\langle \chi, R^{\mathbf{G}}_{\mathbf{T}}(\widehat{s})\rangle_G = \langle \widetilde{\chi}, R^{\widetilde{\mathbf{G}}}_{\widetilde{\mathbf{T}}}(\mathrm{Ind}^{\widetilde{T}}_T(\widehat{s})\rangle_{\widetilde{G}} = \langle \widetilde{\chi}, R^{\widetilde{\mathbf{G}}}_{\widetilde{\mathbf{T}}}(\widehat{\widetilde{s}})\rangle_{\widetilde{G}}. \tag{4.1}$$

Now let $\widetilde{\psi} \in \mathcal{E}(\mathrm{C}_{\widetilde{G}^*}(\widetilde{s}), 1)$, and $\psi \in \mathcal{E}(\mathrm{C}_{G^*}(s), 1)$ with $\widetilde{\psi} = \psi \circ \iota^* = {}^\top\iota^*(\psi)$. Since ι^* restricts to a surjective isotypy from $\mathrm{C}_{\widetilde{G}^*}(\widetilde{s})$ to $\mathrm{C}_{G^*}(s)$ (since $A(s)^F = 1$), and from $\widetilde{T}^*$ to T^*, we have $R^{\mathrm{C}_{\mathbf{G}^*}(s)}_{\mathbf{T}^*}(\mathbb{1}) \circ \iota^* = R^{\mathrm{C}_{\widetilde{\mathbf{G}}^*}(\widetilde{s})}_{\widetilde{\mathbf{T}}^*}(\mathbb{1})$, also by [2, Cor. 2.1.3]. Then

$$\langle \widetilde{\psi}, R^{\mathrm{C}_{\widetilde{\mathbf{G}}^*}(\widetilde{s})}_{\widetilde{\mathbf{T}}^*}(\mathbb{1})\rangle_{\mathrm{C}_{\widetilde{G}^*}(\widetilde{s})} = \langle \psi \circ \iota^*, R^{\mathrm{C}_{\mathbf{G}^*}(s)}_{\mathbf{T}^*}(\mathbb{1}) \circ \iota^*\rangle_{\mathrm{C}_{\widetilde{G}^*}(\widetilde{s})}.$$

Since the kernel of ι^* is central, on which unipotent characters act trivially, we have

$$\langle \psi \circ \iota^*, R^{\mathrm{C}_{\mathbf{G}^*}(s)}_{\mathbf{T}^*}(\mathbb{1}) \circ \iota^*\rangle_{\mathrm{C}_{\widetilde{G}^*}(\widetilde{s})} = \langle \psi, R^{\mathrm{C}_{\mathbf{G}^*}(s)}_{\mathbf{T}^*}(\mathbb{1})\rangle_{\mathrm{C}_{G^*}(s)}.$$

Finally, by either a computation of ranks, or noting that both signs come from the length of the same element of the identified Weyl group, we have $\epsilon_{\widetilde{\mathbf{G}}}\epsilon_{\mathrm{C}_{\widetilde{\mathbf{G}}^*}(\widetilde{s})} = \epsilon_{\mathbf{G}}\epsilon_{\mathrm{C}^\circ_{\mathbf{G}^*}(s)}$. That is,

$$\langle \widetilde{\psi}, \epsilon_{\widetilde{\mathbf{G}}}\epsilon_{\mathrm{C}_{\widetilde{\mathbf{G}}^*}(\widetilde{s})} R_{\widetilde{\mathbf{T}}^*}^{\mathrm{C}_{\widetilde{\mathbf{G}}^*}(\widetilde{s})}(\mathbb{1})\rangle_{\mathrm{C}_{\widetilde{G}^*}(\widetilde{s})} = \langle \psi, \epsilon_{\mathbf{G}}\epsilon_{\mathrm{C}^\circ_{\mathbf{G}^*}(s)} R_{\mathbf{T}^*}^{\mathrm{C}_{\mathbf{G}^*}(s)}(\mathbb{1})\rangle_{\mathrm{C}_{G^*}(s)}. \tag{4.2}$$

Since $J_{\widetilde{s}}$ is a Jordan decomposition map, then (4.1) and (4.2) give the result when taking $\widetilde{\psi} = J_{\widetilde{s}}(\widetilde{\chi})$ so that $\psi = J_{s,\iota}(\chi)$. ☐

5. Toward Unique Jordan Decomposition in Case $A(s)^F = 1$

The following is a list of conditions along the lines of those in Theorem 2.1, but adjusted for a more general situation.

Condition 5.1. If $\mathcal{D} = ((\mathbf{G}, F), (\mathbf{G}^*, F^*), \mathscr{T})$ is a rational duality, then we denote by $\mathcal{S}_{\mathcal{D}} \subseteq G^*$ the set of semisimple elements with $\mathrm{C}_{G^*}(s) \leqslant \mathrm{C}^\circ_{\mathbf{G}^*}(s)$. For each $\mathcal{D}$ and set of bijections

$$J_s^{\mathbf{G}} : \mathcal{E}(G, s) \longrightarrow \mathcal{E}(\mathrm{C}_{G^*}(s), 1),$$

indexed by $s \in \mathcal{S}_{\mathcal{D}}$, we define the following conditions:

1. If $\mathbf{T} \leqslant \mathbf{G}$ and $\mathbf{T}^* \leqslant \mathbf{G}^*$ are dual maximal tori with $\mathbf{T}^* \leqslant \mathrm{C}_{\mathbf{G}^*}(s)$ then for any $\chi \in \mathcal{E}(G, s)$ we have
$$\langle \chi, R_{\mathbf{T}}^{\mathbf{G}}(\widehat{s})\rangle = \langle J_s^{\mathbf{G}}(\chi), \epsilon_{\mathbf{G}}\epsilon_{\mathrm{C}_{\mathbf{G}^*}(s)} R_{\mathbf{T}^*}^{\mathrm{C}_{\mathbf{G}^*}(s)}(\mathbb{1})\rangle.$$
2. If $s = 1$ then:
   (a) The eigenvalues of F^d associated to χ are equal, up to an integer power of q^d, to the eigenvalues of F^{*d} associated to $J_1^{\mathbf{G}}(\chi)$.
   (b) If χ is in the principal series, then $J_1^{\mathbf{G}}(\chi)$ and χ correspond to the same character of the Hecke algebra $\mathrm{End}_{\mathbb{C}G}(\mathrm{Ind}_B^G(\mathbb{1}))$, where $B = \mathbf{B}^F$, and $\mathbf{B}$ is an F-stable Borel subgroup of $\mathbf{G}$.
3. If $z \in \bar{\iota}^*(\mathrm{Z}(\widetilde{G}^*))$ and $\chi \in \mathcal{E}(G, s)$, then $J_{sz}^{\mathbf{G}}(\chi \otimes \widehat{z}) = J_s^{\mathbf{G}}(\chi)$. Here $\bar{\iota}\colon \mathbf{G} \to \widetilde{\mathbf{G}}$ is the specific regular embedding defined below in Definition 8.1.
4. If $\mathbf{L} \leqslant \mathbf{G}$ and $\mathbf{L}^* \leqslant \mathbf{G}^*$ are dual Levi subgroups with $\mathrm{C}^\circ_{\mathbf{G}^*}(s) \leqslant \mathbf{L}^*$ then the following diagram is commutative:
$$\begin{array}{ccc} \mathbb{Z}\mathcal{E}(G, s) & \xrightarrow{J_s^{\mathbf{G}}} & \mathbb{Z}\mathcal{E}(\mathrm{C}_{G^*}(s), 1) \\ \uparrow R_{\mathbf{L}}^{\mathbf{G}} & & \| \\ \mathbb{Z}\mathcal{E}(L, s) & \xrightarrow{J_s^{\mathbf{L}}} & \mathbb{Z}\mathcal{E}(\mathrm{C}_{L^*}(s), 1) \end{array}$$
where $L := \mathbf{L}^F$ and $L^* := \mathbf{L}^{*F^*}$, and we extend J_s by linearity to class functions.
5. Property (5) of Theorem 2.1. (Note: this holds vacuously if $\mathrm{Z}(\mathbf{G}) \neq \mathrm{Z}(\mathbf{G})^\circ$.)
6. Given a surjective isotypy $\varphi : (\mathbf{G}, F) \to (\mathbf{G}_1, F_1)$ such that $\varphi = \widetilde{\varphi} \circ \bar{\iota}$, where $\bar{\iota}\colon \mathbf{G} \to \widetilde{\mathbf{G}}$ is the specific regular embedding defined below in Definition 8.1; $\widetilde{\varphi} : (\widetilde{\mathbf{G}}, F) \to (\widetilde{\mathbf{G}}_1, F_1)$ is an epimorphism with $\ker(\widetilde{\varphi})$ a central torus; and $\mathbf{G}_1 := \widetilde{\varphi}(\bar{\iota}(\mathbf{G}))$ and $\iota_1\colon \mathbf{G}_1 \to \widetilde{\mathbf{G}}_1$ are as in Lemma 8.4, and given semisimple elements $s_1 \in \mathbf{G}_1^*$, $s = \varphi^*(s_1) \in \mathbf{G}^*$, the maps in the following diagram are well-defined and the diagram is commutative:
$$\begin{array}{ccc} \mathbb{Z}\mathcal{E}(G, s) & \xrightarrow{J_s^{\mathbf{G}}} & \mathbb{Z}\mathcal{E}(\mathrm{C}_{G^*}(s), 1) \\ \uparrow {}^\top\varphi & & \downarrow {}^\top\varphi^* \\ \mathbb{Z}\mathcal{E}(G_1, s_1) & \xrightarrow{J_{s_1}^{\mathbf{G}_1}} & \mathbb{Z}\mathcal{E}(\mathrm{C}_{G_1^*}(s_1), 1). \end{array}$$
7. If $\mathbf{G}$ is a direct product $\mathbf{G} = \prod_i \mathbf{G}_i$ of F-stable subgroups, then $J_{\prod_i s_i}^{\mathbf{G}} = \prod_i J_{s_i}^{\mathbf{G}_i}$.

In the next two subsections and in Section 8, we continue to keep the notation from the previous sections and we prove three propositions that will be key to our proof of Theorem 1.1. We remark that in Sections 5.1 and 5.2, which deal with properties (1)-(5), we work without any assumption on the choice of regular embedding $\iota\colon \mathbf{G} \to \widetilde{\mathbf{G}}$. On the other hand, in Section 8, which deals with (6) and (7), we will fix a specific regular embedding $\bar{\iota}$.

5.1. **On Properties (1)-(3).** We begin by considering properties (1)-(3) of Condition 5.1 and Theorem 2.1.

Proposition 5.2. *Let $s \in G^*$ be a semisimple element and suppose that $A(s)^F = 1$. Let $\iota\colon \mathbf{G} \hookrightarrow \widetilde{\mathbf{G}}$ be a regular embedding. Given bijections*

$$f_{s,\iota}\colon \mathcal{E}(G, s) \to \mathcal{E}(\mathrm{C}_{G^*}(s), 1)$$

and

$$f_{\widetilde{s}}\colon \mathcal{E}(\widetilde{G}, s) \to \mathcal{E}(\mathrm{C}_{\widetilde{G}^*}(\widetilde{s}), 1)$$

such that $f_{\widetilde{s}} = {}^{\top}\iota^* \circ f_{s,\iota} \circ \mathrm{Res}^{\widetilde{G}}_{G}$ for each $\widetilde{s} \in \widetilde{G}^*$ such that $\iota^*(\widetilde{s}) = s$, we have $f_{s,\iota}$ satisfies properties (1)-(3) of Condition 5.1 if and only if the collection of $f_{\widetilde{s}}$ satisfies properties (1)-(3) of Theorem 2.1.

We prove Proposition 5.2 by proving the statement for each condition individually.

Lemma 5.3. *Keep the hypotheses of Proposition 5.2. Then Property (1) of Condition 5.1 holds for $f_{s,\iota}$ if and only if Property (1) of Theorem 2.1 holds for $f_{\widetilde{s}}$.*

Proof. Let $\widetilde{\chi} \in \mathcal{E}(\widetilde{G}, \widetilde{s})$ with $\chi = \mathrm{Res}^{\widetilde{G}}_{G}(\widetilde{\chi})$. Also let $\mathbf{T} \leqslant \mathbf{G}$ and $\mathbf{T}^* \leqslant \mathbf{G}^*$ be dual maximal tori with $\mathbf{T}^* \leqslant \mathrm{C}_{\mathbf{G}^*}(s)$, with $\widetilde{\mathbf{T}} \leqslant \widetilde{\mathbf{G}}$ and $\widetilde{\mathbf{T}}^* \leqslant \widetilde{\mathbf{G}}^*$ dual maximal tori with $\mathbf{T} \leqslant \widetilde{\mathbf{T}}$. By (4.1) and (4.2), we have

$$\langle \chi, R^{\mathbf{G}}_{\mathbf{T}}(\widehat{s})\rangle_G = \langle \widetilde{\chi}, R^{\widetilde{\mathbf{G}}}_{\widetilde{\mathbf{T}}}(\widehat{\widetilde{s}})\rangle_{\widetilde{G}}, \quad \text{and}$$

$$\langle f_{\widetilde{s}}(\widetilde{\chi}), \epsilon_{\widetilde{\mathbf{G}}}\epsilon_{\mathrm{C}_{\widetilde{\mathbf{G}}^*}(\widetilde{s})} R^{\mathrm{C}_{\widetilde{\mathbf{G}}^*}(\widetilde{s})}_{\widetilde{\mathbf{T}}^*}(\mathbb{1})\rangle_{\mathrm{C}_{\widetilde{G}^*}(\widetilde{s})} = \langle f_{s,\iota}(\chi), \epsilon_{\mathbf{G}}\epsilon_{\mathrm{C}^\circ_{\mathbf{G}^*}(s)} R^{\mathrm{C}_{\mathbf{G}^*}(s)}_{\mathbf{T}^*}(\mathbb{1})\rangle_{\mathrm{C}^\circ_{G^*}(s)},$$

from which the result follows. ∎

Lemma 5.4. *Keep the hypotheses of Proposition 5.2. Then Property (2) of Condition 5.1 holds for $f_{s,\iota}$ if and only if Property (2) of Theorem 2.1 holds for $f_{\widetilde{s}}$.*

Proof. Let $\widetilde{\chi} \in \mathcal{E}(\widetilde{G}, 1)$, $\chi = \mathrm{Res}^{\widetilde{G}}_{G}(\widetilde{\chi}) \in \mathcal{E}(G, 1)$, $\psi = f_{1,\iota}(\chi) \in \mathcal{E}(G^*, 1)$, and $\widetilde{\psi} = {}^{\top}\iota^*(\psi) = f_1(\widetilde{\chi})$. Consider Condition (2a). Both $\iota : G \to \widetilde{G}$ and $\iota^* : \widetilde{G}^* \to G^*$ are isotypies, which yield our natural bijections of unipotent characters. By arguments in [12, (1.18)], together with [7, Prop. 8.1.13], these bijections of unipotent characters from isotypies preserve the corresponding eigenvalues of the Frobenius, so that the eigenvalues corresponding to χ and $\widetilde{\chi}$ are equal, and the eigenvalues corresponding to ψ and $\widetilde{\psi}$ are equal. The claim for Condition (2a) follows.

For Property (2b), we now assume $\widetilde{\chi}$ is in the principal series, and from which it follows that so are χ, ψ, and $\widetilde{\psi}$. Let $\mathbf{B}$ be an F-stable Borel subgroup of $\mathbf{G}$, and $B = \mathbf{B}^F$. The Hecke algebra for $\widetilde{G}$ (or for G, G^*, or $\widetilde{G}^*$) may be described as $e\mathbb{C}Ge$, with $e = \frac{1}{|B|}\sum_{b\in B} b$, and the bijection from characters in the principal series $\mathrm{Ind}^G_B(1)$ to characters of $e\mathbb{C}Ge$ is given by extending χ from G to $\mathbb{C}G$ linearly, and restricting to the subalgebra $e\mathbb{C}Ge$. Thus the natural bijection between the characters of Hecke algebras corresponding to $\widetilde{G}$ and G is again through restriction, and the natural bijection between the characters of Hecke algebras corresponding to $\widetilde{G}^*$ and G^* is through composition with ι^*, and then extending linearly. That is, $\widetilde{\chi}$ and χ correspond to the same character of the identified Hecke algebras through this bijection, as do $\widetilde{\psi}$ and ψ. These identifications commute with the canonical bijection between the Hecke algebras corresponding to G and G^*, which depends only on the underlying Weyl groups, the duality between Weyl groups W and W^*, and the canonical map of Lusztig between the Weyl group and the Hecke algebra, see [13]. The claim follows. □

Lemma 5.5. *Keep the hypotheses of Proposition 5.2. Then Property (3) of Condition 5.1 (with $\bar\iota$ replaced with the general ι) holds for $f_{s,\iota}$ if and only if Property (3) of Theorem 2.1 holds for the collection of $f_{\widetilde{s}}$.*

Proof. To ease notation, write $f_s := f_{s,\iota}$. First, suppose that Property (3) of Theorem 2.1 holds for $f_{\widetilde{s}}$. Let $z \in \iota^*(\mathrm{Z}(\widetilde{G}^*))$ and $\widetilde{z} \in \mathrm{Z}(\widetilde{G}^*)$ such that $\iota^*(\widetilde{z}) = z$. Then for each $\widetilde{\chi} \in \mathcal{E}(\widetilde{G}, \widetilde{s})$, we have $f_{\widetilde{s}\widetilde{z}}(\widetilde{\chi} \otimes \hat{\widetilde{z}}) = f_{\widetilde{s}}(\widetilde{\chi})$. In particular, let $\chi \in \mathcal{E}(G, s)$ and let $\widetilde{\chi} \in \mathcal{E}(\widetilde{G}, \widetilde{s})$ lie above χ. Note that $\iota^*(\widetilde{s}\widetilde{z}) = sz$. Then by the proof of Lemma 4.1, we may write $f_{sz} = I_s \circ f_{\widetilde{s}\widetilde{z}} \circ \Upsilon_{sz}$, where Υ_{sz} denotes the inverse of the restriction map $\mathrm{Res}^{\widetilde{G}}_G\colon \mathcal{E}(\widetilde{G}, \widetilde{s}\widetilde{z}) \to \mathcal{E}(G, sz)$ and I_s denotes the inverse of the inflation map ${}^\top\iota^*\colon \mathcal{E}(\mathrm{C}_{G^*}(s), 1) \to \mathcal{E}(\mathrm{C}_{\widetilde{G}^*}(\widetilde{s}), 1)$. Further, note that $\Upsilon_{sz}(\chi \otimes \hat{z}) = \widetilde{\chi} \otimes \hat{\widetilde{z}}$, applying Lemma 3.1, since $\widetilde{\chi} \otimes \hat{\widetilde{z}}$ lies above $\chi \otimes \hat{z}$ and is a member of $\mathcal{E}(\widetilde{G}, \widetilde{s}\widetilde{z})$. Hence, we have $f_{sz}(\chi \otimes \hat{z}) = I_s\left(f_{\widetilde{s}\widetilde{z}}(\widetilde{\chi} \otimes \hat{\widetilde{z}})\right) = I_s\left(f_{\widetilde{s}}(\widetilde{\chi})\right) = f_s(\chi)$, as desired.

Conversely, assume that Property (3) of Condition 5.1 holds for f_s. Let $\widetilde{z} \in \mathrm{Z}(\widetilde{G}^*)$ and $\widetilde{\chi} \in \mathcal{E}(\widetilde{G}, \widetilde{s})$. We may now write $f_{\widetilde{s}\widetilde{z}}$ as ${}^\top\iota^* \circ f_{sz} \circ \mathrm{Res}^{\widetilde{G}}_G$ for $z := \iota^*(\widetilde{z})$. Then for $\chi := \mathrm{Res}^{\widetilde{G}}_G(\widetilde{\chi})$, we have $f_{\widetilde{s}\widetilde{z}}(\widetilde{\chi} \otimes \hat{\widetilde{z}}) = {}^\top\iota^* \circ f_{sz}(\chi \otimes \hat{z}) = {}^\top\iota^* \circ f_s(\chi) = f_{\widetilde{s}}(\widetilde{\chi})$, completing the proof. □

5.2. **On Properties (4) and (5).** Here we want to keep the hypotheses from Proposition 5.2, and consider the situation of Properties (4) and (5). Note that by [3, Prop. 2.3 and the preceeding discussion], $\mathrm{C}^\circ_{\mathbf{G}^*}(s) \subseteq \mathbf{L}^*$ for some F^*-stable Levi subgroup $\mathbf{L}^*$ of $\mathbf{G}^*$ if and only if $\mathrm{C}_{\widetilde{\mathbf{G}}^*}(\widetilde{s}) \subseteq \widetilde{\mathbf{L}}^*$ for some F^*-stable Levi subgroup $\widetilde{\mathbf{L}}^*$ of $\widetilde{\mathbf{G}}^*$, where $\iota^*(\widetilde{\mathbf{L}}^*) = \mathbf{L}^*$. Let $\mathbf{L} \leqslant \widetilde{\mathbf{L}}$ be the dual Levi subgroups of $\mathbf{G}$ and $\widetilde{\mathbf{G}}$, and write $L := \mathbf{L}^F$ and $\widetilde{L} := \widetilde{\mathbf{L}}^F$. The condition $A_{\mathbf{G}}(s)^F = 1$ implies that also $A_{\mathbf{L}}(s)^F = 1$, see [17, 1.4], so we obtain a bijection $\mathrm{Res}^{\widetilde{L}}_L\colon \mathcal{E}(\widetilde{L}, \widetilde{s}) \to \mathcal{E}(L, s)$ by applying Lemma 3.2. We may therefore further consider bijections

$$f^L_{s,\iota}\colon \mathcal{E}(L, s) \to \mathcal{E}(\mathrm{C}_{L^*}(s), 1)$$

and

$$f^{\widetilde{L}}_{\widetilde{s}}\colon \mathcal{E}(\widetilde{L}, s) \to \mathcal{E}(\mathrm{C}_{\widetilde{L}^*}(\widetilde{s}), 1)$$

such that $f^{\widetilde{L}}_{\widetilde{s}} = {}^\top\iota^* \circ f^L_{s,\iota} \circ \mathrm{Res}^{\widetilde{L}}_L$.

Proposition 5.6. *Keep the hypotheses of Proposition 5.2. Then, keeping the considerations above, Property (4) (respectively (5)) of Condition 5.1 holds for $f_{s,\iota}$ and $f^L_{s,\iota}$ if and only if Property (4) (respectively (5)) of Theorem 2.1 holds for $f_{\widetilde{s}}$ and $f^{\widetilde{L}}_{\widetilde{s}}$.*

Proof. Again to ease notation, write $f_s := f_{s,\iota}$ and $f^L_s := f^L_{s,\iota}$. Consider the 3D-diagram whose Bottom and Top, respectively, are:

$$\begin{array}{ccc} \mathcal{E}(G, s) & \xrightarrow{f_s} & \mathcal{E}(\mathrm{C}_{G^*}(s), 1) \\ \uparrow R^{\mathbf{G}}_{\mathbf{L}} & & \Vert \\ \mathcal{E}(L, s) & \xrightarrow{f^L_s} & \mathcal{E}(\mathrm{C}_{L^*}(s), 1) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{E}(\widetilde{G}, \widetilde{s}) & \xrightarrow{f_{\widetilde{s}}} & \mathcal{E}(\mathrm{C}_{\widetilde{G}^*}(\widetilde{s}), 1) \\ \uparrow R^{\widetilde{\mathbf{G}}}_{\widetilde{\mathbf{L}}} & & \Vert \\ \mathcal{E}(\widetilde{L}, \widetilde{s}) & \xrightarrow{f^{\widetilde{L}}_{\widetilde{s}}} & \mathcal{E}(\mathrm{C}_{\widetilde{L}^*}(\widetilde{s}), 1) \end{array}$$

Back and Front, respectively, are:

$$\begin{array}{ccc} \mathcal{E}(\widetilde{G},\widetilde{s}) & \xrightarrow{f_{\widetilde{s}}} & \mathcal{E}(\mathrm{C}_{\widetilde{G}^*}(\widetilde{s}),1) \\ \big\downarrow \mathrm{Res}_G^{\widetilde{G}} & & \big\uparrow {}^\top\iota^* \\ \mathcal{E}(G,s) & \xrightarrow{f_s} & \mathcal{E}(\mathrm{C}_{G^*}(s),1) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{E}(\widetilde{L},\widetilde{s}) & \xrightarrow{f_{\widetilde{s}}^{\widetilde{L}}} & \mathcal{E}(\mathrm{C}_{\widetilde{L}^*}(\widetilde{s}),1) \\ \big\downarrow \mathrm{Res}_L^{\widetilde{L}} & & \big\uparrow {}^\top\iota^* \\ \mathcal{E}(L,s) & \xrightarrow{f_s^L} & \mathcal{E}(\mathrm{C}_{L^*}(s),1) \end{array}$$

and whose Left and Right, respectively, are:

$$\begin{array}{ccc} \mathcal{E}(\widetilde{L},\widetilde{s}) & \xrightarrow{R_{\widetilde{\mathbf{L}}}^{\widetilde{\mathbf{G}}}} & \mathcal{E}(\widetilde{G},\widetilde{s}) \\ \big\downarrow \mathrm{Res}_L^{\widetilde{L}} & & \big\downarrow \mathrm{Res}_G^{\widetilde{G}} \\ \mathcal{E}(L,s) & \xrightarrow{R_{\mathbf{L}}^{\mathbf{G}}} & \mathcal{E}(G,s) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{E}(\mathrm{C}_{\widetilde{G}^*}(\widetilde{s}),1) & \xrightarrow{=} & \mathcal{E}(\mathrm{C}_{\widetilde{L}^*}(\widetilde{s}),1) \\ \big\uparrow {}^\top\iota^* & & \big\uparrow {}^\top\iota^* \\ \mathcal{E}(\mathrm{C}_{G^*}(s),1) & \xrightarrow{=} & \mathcal{E}(\mathrm{C}_{L^*}(s),1) \end{array}$$

The Back and Front diagrams commute by our assumption on f_s, $f_{\widetilde{s}}$, f_s^L, and $f_{\widetilde{s}}^{\widetilde{L}}$. The Left diagram commutes because restriction through a regular embedding commutes with twisted induction (see, e.g. [9, Cor. 3.3.25]). The Right diagram commutes because of the equalities. Thus, the Top diagram commutes if and only if the Bottom diagram commutes, completing the proof for property (4).

The case of property (5) has a similar but simpler proof as the above, by noting that the natural bijection between $\mathcal{E}(L,1)$ and $\mathcal{E}(L^*,1)$ preserves cuspidal elements. □

6. Multiplicity Free Restrictions

In this section, we use a difficult result of Lusztig on spin groups to show that restriction from G to $\mathrm{O}^{p'}(G)$ is multiplicity free. From this, we conclude Lusztig's multiplicity freeness result [15, Prop. 10]. Namely, this says that if $\mathbf{G} \to \widetilde{\mathbf{G}}$ is a regular embedding, then restriction from $\widetilde{G} = \widetilde{\mathbf{G}}^F$ to G is multiplicity free.

We note that Li [11, Lem. 2.1] has shown that the restriction map $\mathrm{Res}_{[G,G]}^G$ is multiplicity free assuming "q is large enough". From the proof in [11] we see this is meant to mean that $\mathrm{O}^{p'}(G) = [G,G]$. The proof in [11, Lem. 2.1] relies on [15, Prop. 10], whereas our proof does not utilise regular embeddings at all. We believe the approach taken here may be of interest for other reductions in the future.

As in Lusztig's original approach [15], we need to reduce to the case where $\mathbf{G}$ is simple and simply connected. It is not enough to prove the statement in this case (which is trivial because $G = \mathrm{O}^{p'}(G)$ when $\mathbf{G}$ is simply connected), so we need to prove a different statement. For this we wish to consider finite overgroups of G contained in the normalizer $\mathrm{N}_{\mathbf{G}}(G)$.

If $g \in \mathbf{G}$ satisfies $\mathscr{L}(g) \in \mathrm{Z}(\mathbf{G})$, i.e., $g \in \mathscr{L}^{-1}(\mathrm{Z}(\mathbf{G}))$, then $\mathscr{L}(g)^{-1} = \mathscr{L}(g^{-1})$ and for any $x \in G$ we have

$$F(gxg^{-1}) = \mathscr{L}(g)^{-1}(gF(x)g^{-1})\mathscr{L}(g) = gxg^{-1}$$

so $g \in \mathrm{N}_{\mathbf{G}}(G)$. It is well known that the centralizer $\mathrm{C}_{\mathbf{G}}(G) = \mathrm{Z}(\mathbf{G})$ is the center of $\mathbf{G}$, see [2, Lem. 6.1]. The normalizer may also be described in terms of the center.

Lemma 6.1. *We have* $\mathrm{N}_{\mathbf{G}}(G) = \mathscr{L}^{-1}(\mathrm{Z}(\mathbf{G}))$.

Proof. If $g \in \mathrm{N}_{\mathbf{G}}(G)$ then for any $x \in G$ we have $gxg^{-1} \in G$ so $\mathscr{L}(g) \in \mathrm{C}_{\mathbf{G}}(x)$. Therefore $\mathscr{L}(g) \in \mathrm{C}_{\mathbf{G}}(G) = \mathrm{Z}(\mathbf{G})$ so $\mathrm{N}_{\mathbf{G}}(G) \leqslant \mathscr{L}^{-1}(\mathrm{Z}(\mathbf{G}))$. That $\mathscr{L}^{-1}(\mathrm{Z}(\mathbf{G})) \leqslant \mathrm{N}_{\mathbf{G}}(G)$ is clear. □

Let us draw some conclusions of this equality. Firstly, the natural map $\mathrm{N}_{\mathbf{G}}(G)/\mathrm{C}_{\mathbf{G}}(G) \to (\mathbf{G}/\mathrm{Z}(\mathbf{G}))^F$ is an isomorphism, hence the automizer $\mathrm{N}_{\mathbf{G}}(G)/\mathrm{C}_{\mathbf{G}}(G)$ is a finite group. The image of the natural map $\mathrm{N}_{\mathbf{G}}(G) \to \mathrm{Aut}(G)$ is the group of inner diagonal automorphisms, as defined in [7, §11.5].

Recall that $\mathbf{G} = \mathbf{G}_{\mathrm{der}} \cdot \mathrm{Z}^{\circ}(\mathbf{G})$, where $\mathbf{G}_{\mathrm{der}} \leqslant \mathbf{G}$ is the derived subgroup of $\mathbf{G}$. If $\mathrm{N}_{\mathbf{G}_{\mathrm{der}}}(G) := \mathbf{G}_{\mathrm{der}} \cap \mathrm{N}_{\mathbf{G}}(G)$ then we have a natural map $\mathrm{N}_{\mathbf{G}_{\mathrm{der}}}(G) \to \mathrm{N}_{\mathbf{G}}(G)/\mathrm{C}_{\mathbf{G}}(G)$ whose kernel $\mathrm{C}_{\mathbf{G}_{\mathrm{der}}}(G) := \mathbf{G}_{\mathrm{der}} \cap \mathrm{C}_{\mathbf{G}}(G) = \mathrm{Z}(\mathbf{G}_{\mathrm{der}})$ is finite.

Finally, by the Lang–Steinberg Theorem, the Lang map defines an isomorphism of abstract groups $\mathrm{N}_{\mathbf{G}}(G)/G \to \mathrm{Z}(\mathbf{G})$. The image of the subgroup $(G \cdot \mathrm{Z}(\mathbf{G}))/G$ is the image $\mathscr{L}(\mathrm{Z}(\mathbf{G}))$ of the Lang map. Hence, we have an isomorphism

$$\mathrm{N}_{\mathbf{G}}(G)/(G \cdot \mathrm{Z}(\mathbf{G})) \cong \mathrm{Z}(\mathbf{G})/\mathscr{L}(\mathrm{Z}(\mathbf{G})).$$

As G is finite, we see that there is a bijection $A \mapsto \mathscr{L}^{-1}(A)$ between the finite subgroups of $\mathrm{Z}(\mathbf{G})$ and the finite overgroups $X \leqslant \mathrm{N}_{\mathbf{G}}(G)$ of G.

Lemma 6.2. *If $G_{\mathrm{der}} = (\mathbf{G}_{\mathrm{der}})^F$ then the following hold:*

(i) $\mathrm{N}_{\mathbf{G}_{\mathrm{der}}}(G) = \mathrm{N}_{\mathbf{G}_{\mathrm{der}}}(G_{\mathrm{der}})$ *is finite and* $\mathrm{N}_{\mathbf{G}}(G) = \mathrm{N}_{\mathbf{G}_{\mathrm{der}}}(G) \cdot \mathrm{C}_{\mathbf{G}}(G)$,

(ii) $\mathscr{L}^{-1}(\mathrm{Z}(\mathbf{G}_{\mathrm{der}})) = G \cdot \mathrm{N}_{\mathbf{G}_{\mathrm{der}}}(G)$ *is a finite overgroup of* G,

(iii) *if $X \leqslant \mathrm{N}_{\mathbf{G}}(G)$ is a finite overgroup of G then $X \leqslant \mathrm{N}_{\mathbf{G}_{\mathrm{der}}}(G) \cdot Z$ for some finite subgroup $Z \leqslant \mathrm{C}_{\mathbf{G}}(G)$.*

Proof. (i). As $\mathrm{Z}(\mathbf{G}_{\mathrm{der}}) \leqslant \mathrm{Z}(\mathbf{G})$ we have by Lemma 6.1 that

$$\mathrm{N}_{\mathbf{G}_{\mathrm{der}}}(G) = \mathbf{G}_{\mathrm{der}} \cap \mathscr{L}^{-1}(\mathrm{Z}(\mathbf{G})) = \mathbf{G}_{\mathrm{der}} \cap \mathscr{L}^{-1}(\mathrm{Z}(\mathbf{G}_{\mathrm{der}})) = \mathrm{N}_{\mathbf{G}_{\mathrm{der}}}(G_{\mathrm{der}})$$

which is clearly finite. That $\mathrm{N}_{\mathbf{G}}(G) = \mathrm{N}_{\mathbf{G}_{\mathrm{der}}}(G) \cdot \mathrm{C}_{\mathbf{G}}(G)$ follows immediately from the fact that $\mathbf{G} = \mathbf{G}_{\mathrm{der}} \cdot \mathrm{Z}^{\circ}(\mathbf{G})$.

(ii). By the Lang–Steinberg Theorem $\mathscr{L}(\mathrm{N}_{\mathbf{G}_{\mathrm{der}}}(G)) = \mathrm{Z}(\mathbf{G}_{\mathrm{der}})$ and the fibre of the Lang map over any point is a coset of the finite group G in $\mathbf{G}$. This gives the equality.

(iii). We may simply take Z to be the inverse image of the finite group $(X \cdot \mathrm{N}_{\mathbf{G}_{\mathrm{der}}}(G))/\mathrm{N}_{\mathbf{G}_{\mathrm{der}}}(G)$ under the natural surjective map $\mathrm{C}_{\mathbf{G}}(G) \twoheadrightarrow \mathrm{N}_{\mathbf{G}}(G)/\mathrm{N}_{\mathbf{G}_{\mathrm{der}}}(G)$. □

It will be useful at several points to have some properties regarding the normalizer $\mathrm{N}_{\mathbf{G}}(G)$ with respect to isotypies. Recall that a homomorphism of algebraic groups $\phi : \mathbf{G} \to \mathbf{G}'$ is an *isotypy* if $\mathbf{G}'_{\mathrm{der}} \leqslant \phi(\mathbf{G})$ and $\ker(\phi) \leqslant \mathrm{Z}(\mathbf{G})$. An isotypy $\phi : (\mathbf{G}, F) \to (\mathbf{G}', F')$ is an isotypy $\phi : \mathbf{G} \to \mathbf{G}'$ satisfying $\phi \circ F = F' \circ \phi$.

Lemma 6.3. *Suppose $\phi : \mathbf{G} \to \mathbf{G}'$ is a homomorphism of algebraic groups. If $\mathbf{G}$ is connected and the intersection $\ker(\phi) \cap \mathbf{G}_{\mathrm{der}} \leqslant \mathrm{Z}(\mathbf{G})$ is finite and central, then*

$$\phi^{-1}(\mathrm{Z}(\mathbf{G}')) \leqslant \mathrm{Z}(\mathbf{G}).$$

Proof. Let $A = \ker(\phi) \cap \mathbf{G}_{\mathrm{der}}$. If $g \in \phi^{-1}(\mathrm{Z}(\mathbf{G}'))$ then we have a map $[g, -] : \mathbf{G} \to A$ where $[g, x] = gxg^{-1}x^{-1}$ is the commutator. If $x, y \in \mathbf{G}$ then

$$[g, xy] = gxyg^{-1}y^{-1}x^{-1} = gxg^{-1}[g, y]x^{-1} = [g, x][g, y]$$

because $A \leqslant \mathrm{Z}(\mathbf{G})$ so $[g, -]$ is a group homomorphism. The kernel of $[g, -]$ is the centralizer $\mathrm{C}_{\mathbf{G}}(g)$ so we have an injective homomorphism $\mathbf{G}/\mathrm{C}_{\mathbf{G}}(g) \to A$. As A is finite so is $\mathbf{G}/\mathrm{C}_{\mathbf{G}}(g)$ but as $\mathbf{G}$ is connected we must have $\mathrm{C}_{\mathbf{G}}(g) = \mathbf{G}$ so $g \in \mathrm{Z}(\mathbf{G})$. □

Lemma 6.4. *If $\phi : (\mathbf{G}, F) \to (\mathbf{G}', F')$ is an isotypy and $G = \mathbf{G}^F$ and $G' = \mathbf{G}'^{F'}$ then the following hold:*

(i) $\phi(\mathbf{G}_{\mathrm{der}}) = \mathbf{G}'_{\mathrm{der}}$,

(ii) $\phi(\mathrm{N}_{\mathbf{G}}(G)) = \mathrm{N}_{\phi(\mathbf{G})}(G')$,

(iii) $\phi(\mathscr{L}^{-1}_{\mathbf{G},F}(\mathrm{Z}(\mathbf{G}_{\mathrm{der}}))) = \mathscr{L}^{-1}_{\phi(\mathbf{G}),F'}(\mathrm{Z}(\mathbf{G}'_{\mathrm{der}}))$,

(iv) $\phi(\mathrm{Z}(\mathbf{G})^F) \leq \mathrm{Z}(\mathbf{G}')^{F'}$.

Proof. (i). As $\mathbf{G}' = \mathbf{G}'_{\text{der}} \cdot \mathrm{Z}^\circ(\mathbf{G}')$ we see that $\mathbf{G}' = \phi(\mathbf{G}) \cdot \mathrm{Z}^\circ(\mathbf{G}')$ and so $\mathbf{G}'_{\text{der}} = \phi(\mathbf{G})_{\text{der}} = \phi(\mathbf{G}_{\text{der}})$.

(ii). Assume $g \in \mathbf{G}$ is such that $\phi(g) \in \mathrm{N}_{\mathbf{G}'}(G')$. Then $\phi(\mathscr{L}_{\mathbf{G},F}(g)) = \mathscr{L}_{\mathbf{G}',F'}(\phi(g)) \in \mathrm{Z}(\mathbf{G}')$ and so $\mathscr{L}_{\mathbf{G},F}(g) \in \mathrm{Z}(\mathbf{G})$ by Lemma 6.3, which shows that $g \in \mathrm{N}_{\mathbf{G}}(G)$.

(iii). Using (i) we may argue exactly as in (ii).

(iv). This follows from the observation $\mathbf{G}' = \phi(\mathbf{G}) \cdot \mathrm{Z}^\circ(\mathbf{G}')$. □

We next consider extendability in the case of semisimple groups.

Proposition 6.5. *If $\mathbf{G} = \mathbf{G}_{\text{der}}$ is semisimple, then any character $\chi \in \mathrm{Irr}(\mathrm{O}^{p'}(G))$ extends to its stabiliser $\mathrm{N}_{\mathbf{G}}(G)_\chi$ in the finite group $\mathrm{N}_{\mathbf{G}}(G)$.*

Proof. Let $\phi : \mathbf{G}_{\text{sc}} \twoheadrightarrow \mathbf{G}$ be a simply connected covering map. As ϕ is a bijection on unipotent elements we see that $\phi(\mathrm{O}^{p'}(G_{\text{sc}})) = \mathrm{O}^{p'}(G)$. Denote by $\psi = \chi \circ \phi \in \mathrm{Irr}(\mathrm{O}^{p'}(G_{\text{sc}}))$ the inflation of χ. By Lemma 6.4 we have $\phi(\mathrm{N}_{\mathbf{G}_{\text{sc}}}(G_{\text{sc}})) = \mathrm{N}_{\mathbf{G}}(G)$ so if ψ extends to its stabiliser $\mathrm{N}_{\mathbf{G}_{\text{sc}}}(G_{\text{sc}})_\psi$ then deflating this extension gives an extension of χ to $\mathrm{N}_{\mathbf{G}}(G)_\chi$.

Therefore, we can assume that $\mathbf{G} = \mathbf{G}_{\text{sc}}$ is simply connected, which means $G = \mathrm{O}^{p'}(G)$ by a theorem of Steinberg [28, Thm 12.4]. Suppose $\mathbf{G} = \mathbf{G}^{(1)} \times \cdots \times \mathbf{G}^{(r)}$ is an F-stable decomposition where $\mathbf{G}^{(i)}$ is a product of quasisimple groups. As $\mathrm{N}_{\mathbf{G}}(G) = \mathrm{N}_{\mathbf{G}^{(1)}}(G^{(1)}) \times \cdots \times \mathrm{N}_{\mathbf{G}^{(r)}}(G^{(r)})$ we may clearly assume that $\mathbf{G} = \mathbf{G}_1 \times \cdots \times \mathbf{G}_n$ is a direct product of quasisimple groups permuted transitively by F.

Let $\pi : \mathbf{G} \twoheadrightarrow \mathbf{G}_1$ be the natural projection map. If $g \in \mathrm{N}_{\mathbf{G}}(G)$ then

$$g\mathrm{Z}(\mathbf{G}) \in (\mathbf{G}/\mathrm{Z}(\mathbf{G}))^F \leqslant (\mathbf{G}/\mathrm{Z}(\mathbf{G}))^{F^n} \cong (\mathbf{G}_1/\mathrm{Z}(\mathbf{G}_1))^{F^n} \times \cdots \times (\mathbf{G}_n/\mathrm{Z}(\mathbf{G}_n))^{F^n}$$

so $\pi(g) \in \mathrm{N}_{\mathbf{G}_1}(G_1)$ where $G_1 = \mathbf{G}_1^{F^n}$. Moreover, if $h \in \mathrm{N}_{\mathbf{G}_1}(G_1)$ then $hF(h) \cdots F^{n-1}(h) \in \mathrm{N}_{\mathbf{G}}(G)$ so π restricts to a surjective homomorphism $\mathrm{N}_{\mathbf{G}}(G) \twoheadrightarrow \mathrm{N}_{\mathbf{G}_1}(G_1)$ which further restricts to an isomorphism $G \xrightarrow{\sim} G_1$. Identifying χ with a character of G_1 we see that if χ extends to $\mathrm{N}_{\mathbf{G}_1}(G_1)_\chi$ then inflating we get an extension of χ to $\mathrm{N}_{\mathbf{G}}(G)$. Hence, we can assume that $\mathbf{G}$ is quasisimple and simply connected.

Now if the quotient $\mathrm{N}_{\mathbf{G}}(G)/G \cong \mathrm{Z}(\mathbf{G})/\mathscr{L}(\mathrm{Z}(\mathbf{G}))$ is cyclic then χ will extend to its stabiliser. This is the case unless F is split, $\mathbf{G} = \mathrm{Spin}_{4n}(\mathbb{F})$ is a spin group, and q is odd. But this very tricky case has been dealt with by a counting argument due to Lusztig. A detailed proof of this statement appears in [4, Thm 5.11] and [16]. □

With this, we can now establish the desired extendibility statement for any finite reductive group.

Theorem 6.6. *If $X \leqslant \mathrm{N}_{\mathbf{G}}(G)$ is a finite overgroup of $\mathrm{O}^{p'}(G)$, then any character $\chi \in \mathrm{Irr}(\mathrm{O}^{p'}(G))$ extends to its stabiliser X_χ.*

Proof. Let $H = \mathrm{N}_{\mathbf{G}_{\text{der}}}(G)$. Then by Lemma 6.2, we have $X \leqslant \hat{G} := HZ$ for some finite subgroup $Z \leqslant \mathrm{C}_{\mathbf{G}}(G) = \mathrm{Z}(\mathbf{G})$. It suffices to show that χ extends to its stabilizer $\hat{G}_\chi$. As Z centralises H, we have the product map $\pi : H_\chi \times Z \twoheadrightarrow \hat{G}_\chi$ is a surjective group homomorphism. By Proposition 6.5, χ has an extension $\hat{\chi} \in \mathrm{Irr}(H_\chi)$ because $H = \mathrm{N}_{\mathbf{G}_{\text{der}}}(G_{\text{der}})$ by Lemma 6.2.

Now $H_\chi \cap Z \leqslant \mathrm{Z}(\hat{G})$ so $\mathrm{Res}^{H_\chi}_{H_\chi \cap Z}(\hat{\chi}) = \hat{\chi}(1)\lambda$ for a unique irreducible character $\lambda \in \mathrm{Irr}(H_\chi \cap Z)$. If $\eta \in \mathrm{Irr}(Z)$ is an extension of λ^{-1}, which exists because Z is abelian, then we obtain an irreducible character $\psi = \hat{\chi} \boxtimes \eta \in \mathrm{Irr}(H_\chi \times Z)$ with $\ker(\pi) \leqslant \ker(\psi)$. Deflating ψ gives an extension of χ to $\hat{G}_\chi$. □

Theorem 6.7. *If $X \leqslant Y \leqslant \mathrm{N}_{\mathbf{G}}(G)$ are finite overgroups of $\mathrm{O}^{p'}(G)$, then restriction from Y to X is multiplicity free and every character $\chi \in \mathrm{Irr}(X)$ extends to its stabiliser Y_χ.*

Proof. Let $N = \mathrm{O}^{p'}(G)$. Any character $\psi \in \mathrm{Irr}(N)$ extends to its stabiliser Y_ψ by Theorem 6.6. As Y/N is abelian, we have by Gallagher's Theorem that $\mathrm{Ind}_N^{Y_\psi}(\psi)$ is multiplicity free, hence so is $\mathrm{Ind}_N^Y(\psi)$ by Clifford's correspondence. Frobenius reciprocity now implies that restriction from Y to

N is multiplicity free and so clearly restriction from Y to X must also be multiplicity free. For the last statement we reverse the argument using Frobenius reciprocity and Clifford's correspondence to conclude that $\mathrm{Ind}_X^Y(\chi)$ and hence $\mathrm{Ind}_X^{Y_\chi}(\chi)$ are multiplicity free. Frobenius reciprocity now shows that χ extends to Y_χ. □

If $\mathbf{G} \to \widetilde{\mathbf{G}}$ is a regular embedding then, identifying $\mathbf{G}$ with its image, we have $\mathrm{O}^{p'}(\widetilde{G}) \leqslant G \leqslant \widetilde{G} \leqslant \mathrm{N}_{\widetilde{\mathbf{G}}}(\widetilde{G})$ so from this we obtain Lusztig's result [15, Prop. 10]. The following shows that the usual information one obtains from a regular embedding can be read off from the finite overgroup $G \cdot \mathrm{N}_{\mathbf{G}_{\mathrm{der}}}(G) \leqslant \mathrm{N}_{\mathbf{G}}(G)$ of G.

Lemma 6.8. *Assume $\iota : (\mathbf{G}, F) \to (\widetilde{\mathbf{G}}, F)$ is an injective isotypy and $\mathrm{Z}(\widetilde{\mathbf{G}})$ is connected. Let $\hat{G} = G \cdot \mathrm{N}_{\mathbf{G}_{\mathrm{der}}}(G)$. For any $\chi \in \mathrm{Irr}(G)$ we have an isomorphism $\widetilde{G}/\widetilde{G}_\chi \cong \hat{G}/\hat{G}_\chi$ where $\widetilde{G} = \widetilde{\mathbf{G}}^F$.*

Proof. We identify $\mathbf{G}$, as an abstract group, with its image in $\widetilde{\mathbf{G}}$. Let $\Gamma = \hat{G}Z$ where $Z = \mathrm{Z}(\widetilde{\mathbf{G}}) \cap \mathscr{L}_{\widetilde{\mathbf{G}},F}^{-1}(\mathrm{Z}(\mathbf{G}_{\mathrm{der}}))$. It suffices to show that $\Gamma = \widetilde{G}Z$ because this implies that

$$\hat{G}/\hat{G}_\chi \cong \Gamma/\Gamma_\chi \cong \widetilde{G}/\widetilde{G}_\chi.$$

Suppose $g \in \hat{G}$ so that $\mathscr{L}(g) \in \mathrm{Z}(\mathbf{G}_{\mathrm{der}})$. As $\mathrm{Z}(\widetilde{\mathbf{G}})$ is connected and contains $\mathrm{Z}(\mathbf{G}_{\mathrm{der}})$ we have by the Lang–Steinberg Theorem that $\mathscr{L}(g) = \mathscr{L}(z)$ for some $z \in \mathrm{Z}(\widetilde{\mathbf{G}})$. Certainly $z \in Z$ and $gz^{-1} \in G$ so $g \in \widetilde{G}Z$. This shows that $\Gamma \leqslant \widetilde{G}Z$. But if $g \in \widetilde{G}$ then $g = hz$ for some $h \in \mathbf{G}_{\mathrm{der}}$ and $z \in \mathrm{Z}(\widetilde{\mathbf{G}})$. As $\mathscr{L}(g) = 1$ we have $\mathscr{L}(h) = \mathscr{L}(z^{-1}) \in \mathbf{G}_{\mathrm{der}} \cap \mathrm{Z}(\widetilde{\mathbf{G}}) = \mathrm{Z}(\mathbf{G}_{\mathrm{der}})$ so $\widetilde{G}Z \leqslant \Gamma$. □

7. Isotypies and Deligne–Lusztig Induction

In this section, we develop further results on isotypies. For this, we will need the following result on Deligne–Lusztig induction, which generalises a standard result found in [7, Prop. 11.3.10]. A related statement on 2-variable Green functions is found in [2, Prop. 2.2.2]. One could prove Proposition 7.1 by reducing to the statement in [2, Prop. 2.2.2]. However, we prefer to give a proof of Proposition 7.1, from which [2, Prop. 2.2.2] is then an easy consequence.

Before stating the result, let us introduce the following notation. If $\mathbf{X}$ is a variety and $g \in \mathrm{Aut}(\mathbf{X})$ is an element of finite order, then we denote by

$$\mathcal{L}(g \mid \mathbf{X}) = \sum_{i\in\mathbb{Z}} (-1)^i \mathrm{Tr}(g \mid H_c^i(\mathbf{X}, \overline{\mathbb{Q}}_\ell))$$

the Lefschetz trace of g acting on the cohomology of $\mathbf{X}$.

Proposition 7.1. *Suppose $\phi : (\mathbf{G}, F) \to (\mathbf{G}', F')$ is an isotypy with kernel $\mathbf{K} = \ker(\phi) \leqslant \mathrm{Z}(\mathbf{G})$ and let $\mathbf{L}' \leqslant \mathbf{G}'$ be an F'-stable Levi complement of a parabolic subgroup $\mathbf{P}' \leqslant \mathbf{G}'$. If $(\mathbf{L}, \mathbf{P}) = (\phi^{-1}(\mathbf{L}'), \phi^{-1}(\mathbf{P}'))$ then*

$$^\top\phi \circ R_{\mathbf{L}'\subseteq\mathbf{P}'}^{\mathbf{G}'} = \frac{1}{|\mathbf{K}/\mathscr{L}(\mathbf{K})|} \sum_{z\mathscr{L}(\mathbf{K})\in\mathbf{K}/\mathscr{L}(\mathbf{K})} R_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}} \circ {}^\top\mathrm{Ad}_{l_z} \circ {}^\top\phi$$

where $l_z \in \mathbf{L}$ is an element satisfying $\mathscr{L}(l_z) = z$.

Proof. Let $\mathbf{U} \leqslant \mathbf{P}$ be the unipotent radical of $\mathbf{P}$ then $\mathbf{U}' = \phi(\mathbf{U})$ is the unipotent radical of $\mathbf{P}'$. We define $\mathbf{Y}' = \{g \in \mathbf{G}' \mid \mathscr{L}(g) \in \mathbf{U}'\}$. If $z \in \mathrm{Z}(\mathbf{G})$ then we let $\mathbf{Y}_z = \{g \in \mathbf{G} \mid \mathscr{L}(g) \in \mathbf{U}z\}$, which is a closed subset of $\mathbf{G}$. Note that $\mathbf{Y}_z \cap \mathbf{Y}_{z'} = \emptyset$ if $z \neq z'$. Fix an element $l_z \in \mathbf{L}$ such that $\mathscr{L}(l_z) = z$. If $g \in \mathbf{Y}_z$ then

$$\mathscr{L}(gl_z^{-1}) = l_z\mathscr{L}(g)l_z^{-1}\mathscr{L}(l_z^{-1}) \in (\mathbf{U}z)z^{-1} = \mathbf{U}$$

because $l(\mathbf{U}z)l^{-1} = \mathbf{U}z$ for any $l \in \mathbf{L}$. This shows that $\mathbf{Y}_z = \mathbf{Y}_1 l_z$.

We choose a finite subgroup $A \leqslant \mathbf{K}$ such that $\mathbf{K} = A\cdot\mathscr{L}(\mathbf{K})$. One can obtain such a subgroup as the group $\langle a_1,\dots,a_r\rangle$ generated by a set of representatives $a_1,\dots,a_r \in \mathbf{K}$ for the cosets $\mathbf{K}/\mathscr{L}(\mathbf{K})$. This is a finite group because $\mathbf{K}$ is abelian, $\mathbf{K}/\mathscr{L}(\mathbf{K})$ is finite, and each a_i has finite order.

As A is finite we have $\mathbf{Y} = \bigsqcup_{a\in A}\mathbf{Y}_a$ is a closed subset of $\mathbf{G}$. If $g \in \mathbf{Y}_a$ for some $a \in A$ then $\mathscr{L}(\phi(g)) = \phi(\mathscr{L}(g)) \in \phi(\mathbf{U}a) = \mathbf{U}'$ so $\phi(\mathbf{Y}) \subseteq \mathbf{Y}'$. We show that $\phi(\mathbf{Y}) = \mathbf{Y}'$. Suppose $h \in \mathbf{Y}'$ so that $u = \mathscr{L}(h) \in \mathbf{U}'$. If $g \in \mathbf{G}$ satisfies $\phi(g) = h$ and $v \in \mathbf{U}$ is the unique element satisfying $\phi(v) = u$ then $\mathscr{L}(g) = vt$ for some $t \in \mathbf{K}$. We can write $t = \mathscr{L}(z)a$ for some $z \in \mathbf{K}$ and $a \in A$. Then $gz^{-1} \in \mathbf{Y}_a$ and $\phi(gz^{-1}) = h$.

Let $\hat{K} = \mathbf{K}\cap\mathscr{L}_{\mathbf{G},F}^{-1}(A)$, which is a finite subgroup of $\mathbf{K}$. We will now show that the fibres of the map $\phi : \mathbf{Y} \twoheadrightarrow \mathbf{Y}'$ are the orbits of $\hat{K}$ acting by translations. Suppose $g,h \in \mathbf{Y}$ satisfy $\phi(g) = \phi(h)$ so that $g = ht$ for some $t \in \mathbf{K}$. We have $g \in \mathbf{Y}_a$ for some $a \in A$ and $h \in \mathbf{Y}_b$ for some $b \in A$. But $\mathscr{L}(g) = \mathscr{L}(h)\mathscr{L}(t) \in \mathbf{U}\mathscr{L}(t)b \cap \mathbf{U}a$ which forces $\mathscr{L}(t)b = a$ and so $\mathscr{L}(t) = ab^{-1} \in A$. Therefore $\phi : \mathbf{Y} \twoheadrightarrow \mathbf{Y}'$ factors through a bijective morphism $\mathbf{Y}/\hat{K} \to \mathbf{Y}'$.

The finite group $\hat{L} = \mathscr{L}_{\mathbf{L},F}^{-1}(A)$ satisfies $\phi(\hat{L}) = L' := \mathbf{L}'^F$ and ϕ factors through an isomorphism $\hat{L}/\hat{K} \to L'$. If $l \in \hat{L}$ and $x \in \mathbf{Y}_a$, with $a \in A$, then $\mathscr{L}(xl) \in \mathbf{U}a\mathscr{L}(l)$ and so $\mathbf{Y}_a l = \mathbf{Y}_{a\mathscr{L}(l)}$. This shows that $\hat{L}$ acts on $\mathbf{Y}$ by right translation. Moreover, $\mathbf{Y}_a l = \mathbf{Y}_a$ if and only if $l \in \mathbf{L}^F$. Therefore, if $G \times \hat{L}^{\mathrm{opp}}$ acts on $\mathbf{Y}$ via the action $(g,l)\cdot x = gxl$ then

$$\begin{aligned}\mathcal{L}((g,l) \mid \mathbf{Y}/\hat{K}) &= \frac{1}{|\hat{K}|}\sum_{\substack{z\in\hat{K}\\ \mathscr{L}(lz)=1}}\sum_{a\in A}\mathcal{L}((g,lz) \mid \mathbf{Y}_a)\\ &= \frac{1}{|\hat{K}|}\sum_{\substack{z\in\hat{K}\\ \mathscr{L}(lz)=1}}\sum_{a\in A}\mathcal{L}((g,l_a^{-1}lzl_a) \mid \mathbf{Y}_1).\end{aligned}$$

Here we use [7, 8.1.10(ii)] and [7, Prop. 8.1.7(iii)] for the first equality, together with the fact that $\mathbf{Y}_a = \mathbf{Y}_1 l_a$ for the second equality.

Let $\pi : \hat{L}\times\hat{K} \to \mathbf{L}$ be the product map given by $\pi(l,z) = lz$. If $g = \pi(l,z)$ then the fibre $\pi^{-1}(g) = \{(lx, x^{-1}z) \mid x \in \hat{K}\}$ is in bijection with $\hat{L}\cap\hat{K} = \hat{K}$. Summing over $\hat{L}$ we get

$$\begin{aligned}\frac{1}{|L|}\sum_{l\in\hat{L}}\mathcal{L}((g,l) \mid \mathbf{Y}/\hat{K}) &= \frac{1}{|L||\hat{K}|}\sum_{\substack{(l,z)\in\hat{L}\times\hat{K}\\ \mathscr{L}(lz)=1}}\sum_{a\in A}\mathcal{L}((g,l_a^{-1}lzl_a) \mid \mathbf{Y}_1)\\ &= \frac{1}{|L|}\sum_{l\in L}\sum_{a\in A}\mathcal{L}((g,l_a^{-1}ll_a) \mid \mathbf{Y}_1)\end{aligned}$$

Now using Lemma 9.1.5 and Proposition 9.1.6 from [7] we see that

$$\begin{aligned}R_{\mathbf{L}'\subseteq\mathbf{P}'}^{\mathbf{G}'}(\chi)(\phi(g)) &= \frac{1}{|L'|}\sum_{l\in\mathbf{L}'^F}\mathcal{L}((\phi(g),l) \mid \mathbf{Y}')\chi(l^{-1})\\ &= \frac{1}{|\hat{L}|}\sum_{l\in\hat{L}}\mathcal{L}((g,l) \mid \mathbf{Y}/\hat{K})\cdot(\chi\circ\phi)(l^{-1})\\ &= \frac{1}{|A|}\sum_{a\in A}\frac{1}{|L|}\sum_{l\in L}\mathcal{L}((g,l) \mid \mathbf{Y}_1)\cdot(\chi\circ\phi)((l_a l l_a^{-1})^{-1})\\ &= \frac{1}{|A|}\sum_{a\in A}R_{\mathbf{L}}^{\mathbf{G}}(\chi\circ\phi\circ\mathrm{Ad}_{l_a})(g)\end{aligned}$$

If $a \in A$ is contained in the kernel of the map $A \to \mathbf{K}/\mathscr{L}(K)$ then Ad_{l_a} restricts to an inner automorphism of L. Hence, we may take the sum over $A/(A\cap\mathscr{L}(\mathbf{K})) \cong \mathbf{K}/\mathscr{L}(\mathbf{K})$. □

Corollary 7.2. *If* $\mathbf{K} \leqslant \mathscr{L}(\mathrm{Z}(\mathbf{L}))$ *in the setting of Proposition 7.1, then* ${}^{\top}\phi \circ R_{\mathbf{L}'}^{\mathbf{G}'} = R_{\mathbf{L}}^{\mathbf{G}} \circ {}^{\top}\phi$*. This condition is satisfied if either* $\mathbf{K}$ *is connected or* $\mathrm{Z}(\mathbf{L})$ *is connected.*

Proof. Under our assumption, we may choose $l_z \in \mathrm{Z}(\mathbf{L})$ such that $\mathscr{L}(l_z) = z$. In this case $\mathrm{Ad}_{l_z}|_{\mathbf{L}}$ is trivial and the statement follows. Note that $\mathbf{K} \leqslant \mathrm{Z}(\mathbf{G}) \leqslant \mathrm{Z}(\mathbf{L})$ and if $\mathbf{K}$ is connected then $\mathbf{K} = \mathscr{L}(\mathbf{K}) \leqslant \mathscr{L}(\mathrm{Z}(\mathbf{L}))$ and if $\mathrm{Z}(\mathbf{L})$ is connected then $\mathscr{L}(\mathrm{Z}(\mathbf{L})) = \mathrm{Z}(\mathbf{L})$. □

Note that Corollary 7.2 applies in particular to the case where $\mathbf{L} = \mathrm{Z}(\mathbf{L})$ is a torus. We investigate the implications this has for Lusztig series, following the arguments presented in [29, Prop. 7.2]. First we need to extend the discussion of dual isogenies to isotypies. For this, we follow [19, Def. 2.11].

Definition 7.3. Assume $(\mathbf{G}, F)$ and $(\mathbf{G}', F')$ are finite reductive groups with dual groups $(\mathbf{G}^*, F^*)$ and $(\mathbf{G}'^*, F'^*)$ with the dualities witnessed by $(\mathbf{T}_0, \mathbf{T}_0^*, \delta)$ and $(\mathbf{T}_0', \mathbf{T}_0'^*, \delta')$ respectively. Two isotypies $\phi : (\mathbf{G}, F) \to (\mathbf{G}', F')$ and $\phi^* : (\mathbf{G}'^*, F'^*) \to (\mathbf{G}^*, F^*)$ are said to be *dual* if

$$\check{X}(\phi^* \circ \mathrm{Ad}_{g^*}) \circ \delta' = \delta \circ X(\phi \circ \mathrm{Ad}_g) \tag{7.1}$$

for some $(g, g^*) \in \mathbf{G} \times \mathbf{G}'^*$ satisfying $\phi({}^g\mathbf{T}_0) \leqslant \mathbf{T}_0'$ and $\phi^*({}^{g^*}\mathbf{T}_0'^*) \leqslant \mathbf{T}_0^*$.

Note the condition in (7.1) generalises the condition in (2.1). We need the following.

Lemma 7.4. *Any isotypy* $\phi : (\mathbf{G}, F) \to (\mathbf{G}', F')$ *admits a dual* $\phi^* : (\mathbf{G}'^*, F'^*) \to (\mathbf{G}^*, F^*)$*, which is unique up to composing with some* Ad_h *with* $h \in \mathrm{N}_{\mathbf{G}'^*}(G'^*)$*.*

Proof. Note that $\phi(\mathbf{T}_0)$ is a torus so is contained in a maximal torus of $\mathbf{G}'$. By the conjugacy of maximal tori there exists an element $g' \in \mathbf{G}'$ such that ${}^{g'}\phi(\mathbf{T}_0) \leqslant \mathbf{T}_0'$. As $\mathbf{G}' = \mathbf{G}'_{\mathrm{der}} \cdot \mathrm{Z}^\circ(\mathbf{G}')$ we can assume that $g' \in \mathbf{G}'_{\mathrm{der}}$. Using (i) of Lemma 6.4 there is an element $g \in \mathbf{G}_{\mathrm{der}}$ such that $\phi(g) = g'$ and so $\phi({}^g\mathbf{T}_0) \leqslant \mathbf{T}_0'$.

The composition $\tilde{\phi} = \phi \circ \mathrm{Ad}_g$ is an isotypy $(\mathbf{G}, \tilde{F}) \to (\mathbf{G}', F')$, where $\tilde{F} = \mathrm{Ad}_{\mathscr{L}(g)} \circ F$, which satisfies $\tilde{\phi}(\mathbf{T}_0) \leqslant \mathbf{T}_0'$. Using the bijections stated in Section 2 we see that we have a bijection

$$^* : \mathrm{Hom}(\mathbf{T}_0, \mathbf{T}_0') \xrightarrow{\sim} \mathrm{Hom}(\mathbf{T}_0'^*, \mathbf{T}_0^*),$$

which is defined by requiring that $X(f) = \delta^{-1} \circ \check{X}(f^*) \circ \delta'$.

If $\tilde{f} = \tilde{\phi}|_{\mathbf{T}_0}$ then as $\tilde{\phi}$ is an isotypy we have $X(\tilde{f})$ defines a p-morphism of root data as defined in [30, 3.2]. It is easy to see that $\check{X}(\tilde{f}^*)$ will be a p-morphism of root data, because $X(\tilde{f})$ is, and so $X(\tilde{f}^*)$ will be as well. By an extension of the isogeny theorem, see [30, Thm 3.8] and the references therein, there exists an isotypy $\phi^* : \mathbf{G}'^* \to \mathbf{G}^*$ such that $\phi^*(\mathbf{T}_0'^*) \leqslant \mathbf{T}_0^*$ and $\phi^*|_{\mathbf{T}_0'^*} = \tilde{f}^*$. This is clearly dual to ϕ.

We now consider the unicity of ϕ^*. If $h \in \mathrm{N}_{\mathbf{G}'^*}(G'^*)$ and $\psi = \phi^* \circ \mathrm{Ad}_h$ then certainly $F^* \circ \psi = \psi \circ F'^*$ so ψ is an isogeny $(\mathbf{G}'^*, F'^*) \to (\mathbf{G}^*, F^*)$. As $\psi \circ \mathrm{Ad}_{h^{-1}g^*} = \phi^* \circ \mathrm{Ad}_{g^*}$ it follows straightforwardly that ϕ and ψ are dual.

Conversely, suppose ϕ and ψ are dual and let $y \in \mathbf{G}'^*$ be an element such that $\psi({}^y\mathbf{T}_0'^*) \leqslant \mathbf{T}_0^*$. By the conjugacy of maximal tori there exists an element $x \in \mathbf{G}'^*$ such that ${}^{xy}\mathbf{T}_0'^* = {}^{g^*}\mathbf{T}_0'^*$. Hence $\psi' = \psi \circ \mathrm{Ad}_{x^{-1}}$ satisfies $\psi'({}^{g^*}\mathbf{T}_0'^*) \leqslant \mathbf{T}_0^*$. Because ψ and ϕ^* both satisfy (7.1) we must have

$$\check{X}(\psi' \circ \mathrm{Ad}_{xy}) = \check{X}(\psi \circ \mathrm{Ad}_y) = \check{X}(\phi^* \circ \mathrm{Ad}_{g^*}).$$

This implies that $\psi' \circ \mathrm{Ad}_{xyt} = \phi^* \circ \mathrm{Ad}_{g^*}$ for some $t \in \mathbf{T}_0'^*$, see [30, Thm 3.8]. In particular, $\psi = \phi^* \circ \mathrm{Ad}_h$ for some $h \in \mathbf{G}'^*$.

As ψ and ϕ^* both commute with F'^* and F^*, which are bijective, we must have $\phi^*\mathrm{Ad}_{F'^*(h)h^{-1}} = \phi^*$. Hence, there exists a homomorphism $\pi : \mathbf{G}'^* \to \ker(\phi^*) \leqslant \mathrm{Z}(\mathbf{G}'^*)$ such that $\mathrm{Ad}_{F'^*(h)h^{-1}}(x) = x\pi(x)$ for all $x \in \mathbf{G}'^*$. However $\mathbf{G}'^*_{\mathrm{der}} \leqslant \ker(\pi)$, because $\ker(\phi^*)$ is abelian, and $\mathrm{Z}(\mathbf{G}'^*)$ must also be in $\ker(\pi)$ by definition. Therefore π is trivial so $F'^*(h)h^{-1} \in \mathrm{Z}(\mathbf{G}'^*)$ and we may apply Lemma 6.1. □

We can now give the analogue of [29, Prop. 7.2] for arbitrary isotypies.

Proposition 7.5. *Assume $\phi : (\mathbf{G}, F) \to (\mathbf{G}', F')$ is an isotypy and $\phi^* : (\mathbf{G}'^*, F'^*) \to (\mathbf{G}^*, F^*)$ is dual to ϕ. If $\chi' \in \mathcal{E}(\mathbf{G}'^{F'}, s')$, for some $s' \in \mathbf{G}'^{F'}$, then every irreducible constituent of $\phi^\top(\chi')$ is contained in $\mathcal{E}(\mathbf{G}^F, \phi^*(s'))$.*

Proof. Recall that the regular character is uniform [7, Cor. 10.2.6] and projecting that expression onto the subspace spanned by $\mathcal{E}(\mathbf{G}^F, s)$ we see that

$$\sum_{\chi' \in \mathcal{E}(\mathbf{G}'^{F'}, s')} \chi'(1)\chi' = \frac{1}{|\mathbf{G}'^{F'}|_{p'}} \sum_{(\mathbf{T}', \theta') \in \in \nabla_{[s']}(\mathbf{G}', F')} \varepsilon_{\mathbf{G}'} \varepsilon_{\mathbf{T}'} R_{\mathbf{T}'}^{\mathbf{G}'}(\theta').$$

Here $\nabla_{[s']}(\mathbf{G}', F')$ denotes the set of pairs $(\mathbf{T}', \theta')$ that are dual to some $(\mathbf{T}'^*, s')$. Now applying $\phi^\top$ to this expression we get

$$\sum_{\chi' \in \mathcal{E}(\mathbf{G}'^{F'}, s')} \chi'(1)\phi^\top(\chi') = \frac{1}{|\mathbf{G}'^{F'}|_{p'}} \sum_{(\mathbf{T}', \theta') \in \in \nabla_{[s']}(\mathbf{G}', F')} \varepsilon_{\mathbf{G}'} \varepsilon_{\mathbf{T}'} R_{\phi^{-1}(\mathbf{T}')}^{\mathbf{G}}(\phi^\top(\theta')).$$

If $\chi \in \operatorname{Irr}(\mathbf{G}^F)$ is an irreducible constituent of $\phi^\top(\chi')$ then χ occurs with non-zero multiplicity in the left-hand sum. Hence, there must exist a pair $(\mathbf{T}, \theta) = (\phi^{-1}(\mathbf{T}'), \phi^\top(\theta'))$ such that $\phi^\top(\chi')$ is a consituent of $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$.

We now just need to show that if $(\mathbf{T}', \theta')$ corresponds to $(\mathbf{T}'^*, s')$ then $(\mathbf{T}, \theta)$ corresponds to $(\mathbf{T}^*, \phi^*(s'))$. The argument here is exactly the same as that given in the proof of [29, Prop. 7.2], which we note relies only on the property in (7.1). □

The G'^*-conjugacy class of s' and ${}^h s'$ is the same for any $h \in \mathrm{N}_{\mathbf{G}'^*}(G'^*)$. This is because $\mathrm{C}_{\mathbf{G}'^*}(s')$ contains a maximal torus of $\mathbf{G}'^*$, so $\mathrm{Z}(\mathbf{G}'^*)$ is contained in the connected component of $\mathrm{C}_{\mathbf{G}'^*}(s')$. Hence, the series $\mathcal{E}(\mathbf{G}^F, \phi^*(s'))$ is the same regardless of which dual isotypy we pick by Lemma 7.4.

8. On properties (6) and (7)

In this section, we will sometimes want to fix a regular embedding $\bar{\iota} \colon \mathbf{G} \to \widetilde{\mathbf{G}}$ with some nice properties. Throughout, $\bar{\iota}$ will refer to the following regular embedding:

Definition 8.1. Let $(\mathbf{G}, F)$ be a finite reductive group with fixed F-stable maximal torus $\mathbf{T} \leqslant \mathbf{G}$ and define $\widetilde{\mathbf{G}} = (\mathbf{G} \times \mathbf{T})/\Delta$ where

$$\Delta = \{(z, z^{-1}) \mid z \in \mathrm{Z}(\mathbf{G})\}.$$

We have $\widetilde{\mathbf{G}}$ inherits a Frobenius endomorphism, defined by $F((g,t)\Delta) = (F(g), F(t))\Delta$ and the natural map $\bar{\iota} : (\mathbf{G}, F) \to (\widetilde{\mathbf{G}}, F)$, given by $g \mapsto (g, 1)\Delta$, is a regular embedding.

Here we prove the following:

Proposition 8.2. *Assume $((\mathbf{G}, F), (\mathbf{G}^*, F^*), \mathscr{T})$ is a rational duality and let $\bar{\iota} \colon \mathbf{G} \to \widetilde{\mathbf{G}}$ be the particular regular embedding defined in Definition 8.1. For each semisimple $s \in G^*$ with $A_{\mathbf{G}}(s)^F = 1$, let $\widetilde{s} \in \widetilde{G}^*$ be an element such that $\bar{\iota}^*(\widetilde{s}) = s$, and assume we have bijections*

$$f_{s,\bar{\iota}} \colon \mathcal{E}(G, s) \to \mathcal{E}(\mathrm{C}_{G^*}(s), 1)$$

and

$$f_{\widetilde{s}} \colon \mathcal{E}(\widetilde{G}, s) \to \mathcal{E}(\mathrm{C}_{\widetilde{G}^*}(\widetilde{s}), 1)$$

such that $f_{\widetilde{s}} = {}^\top\bar{\iota}^* \circ f_{s,\bar{\iota}} \circ \operatorname{Res}_G^{\widetilde{G}}$. Then we have $f_{s,\bar{\iota}}$ satisfies properties (6) and (7) of Condition 5.1 if and only if $f_{\widetilde{s}}$ satisfies properties (6) and (7) of Theorem 2.1.

We start with property (7).

Lemma 8.3. *Keep the hypotheses of Proposition 8.2. Then Property (7) of Condition 5.1 holds for $f_{s,\bar{\iota}}$ if and only if Property (7) of Theorem 2.1 holds for $f_{\tilde{s}}$.*

Proof. We need only consider when both $\mathbf{G}$ and $\widetilde{\mathbf{G}}$ are direct products with F-stable factors. If $\mathbf{G} = \prod_j \mathbf{G}_j$ is such a direct product, then an F-stable maximal torus of $\mathbf{G}$ is of the form $\prod_j \mathbf{T}_j$ with $\mathbf{T}_j$ an F-stable maximal torus of $\mathbf{G}_j$. Defining $\Delta_j = \{(z, z^{-1}) \mid z \in \mathrm{Z}(\mathbf{G}_j)\}$, we then have

$$\widetilde{\mathbf{G}} = \left(\prod_j \mathbf{G}_j \times \prod_j \mathbf{T}_j\right) / \prod_j \Delta_j \cong \prod_j ((\mathbf{G}_j \times \mathbf{T}_j)/\Delta_j) = \prod_j \widetilde{\mathbf{G}}_j,$$

where $\bar{\iota}_j : \mathbf{G}_j \to \widetilde{\mathbf{G}}_j$ are regular embeddings as in Definition 8.1. That is, the regular embedding $\bar{\iota}$ directly factors as $\bar{\iota} = \prod_j \bar{\iota}_j$. We have $s = \prod_j s_j$ and $\tilde{s} = \prod_j \tilde{s}_j$. Since $f_{\tilde{s}} = {}^{\top}\bar{\iota}^* \circ f_{s,\bar{\iota}} \circ \mathrm{Res}^{\widetilde{G}}_{G}$, where ${}^{\top}\bar{\iota}^*$ factors over the direct product, and $\mathrm{Res}^{\widetilde{G}}_{G}$ naturally factors over the direct product, it follows that $f_{\tilde{s}} = \prod_j f_{\tilde{s}_j}$ if and only if $f_{s,\bar{\iota}} = \prod_j f_{s_j,\bar{\iota}_j}$, as claimed. □

8.1. **On Property (6).** Finally, we move on to discussing Property (6).

Lemma 8.4. *Let $\iota \colon \mathbf{G} \hookrightarrow \widetilde{\mathbf{G}}$ be a regular embedding. Let $\widetilde{\mathbf{G}}_1$ be a connected reductive algebraic group defined over $\mathbb{F}_q$ with Frobenius map F_1 such that $\mathrm{Z}(\widetilde{\mathbf{G}}_1)$ is connected. Given an epimorphism $\widetilde{\varphi} : (\widetilde{\mathbf{G}}, F) \to (\widetilde{\mathbf{G}}_1, F_1)$ such that $\ker(\widetilde{\varphi})$ is central, define $\mathbf{G}_1 := \widetilde{\varphi}(\iota(\mathbf{G}))$. Then the natural embedding $\iota_1 \colon \mathbf{G}_1 \hookrightarrow \widetilde{\mathbf{G}}_1$ is also a regular embedding.*

Proof. Since the image of a morphism of algebraic groups is closed, we see $\mathbf{G}_1$ is a closed subgroup of $\widetilde{\mathbf{G}}_1$. Since $\widetilde{\varphi}$ is surjective, any element of $[\widetilde{\mathbf{G}_1}, \widetilde{\mathbf{G}_1}]$ can be written as a product of elements of the form $[\widetilde{\varphi}(g), \widetilde{\varphi}(h)] = \widetilde{\varphi}([g,h])$ for $g, h \in \widetilde{\mathbf{G}}$. Then $[\widetilde{\mathbf{G}_1}, \widetilde{\mathbf{G}_1}] \leq \widetilde{\varphi}([\widetilde{\mathbf{G}}, \widetilde{\mathbf{G}}]) = \widetilde{\varphi}([\iota(\mathbf{G}), \iota(\mathbf{G})])$ since ι is a regular embedding. Hence $[\widetilde{\mathbf{G}_1}, \widetilde{\mathbf{G}_1}] \leq [\varphi(\mathbf{G}), \varphi(\mathbf{G})] = [\mathbf{G}_1, \mathbf{G}_1]$. □

Lemma 8.5. *Let $s \in G^*$ be semisimple such that $A_{\mathbf{G}}(s)^F = 1$. Let $\varphi : (\mathbf{G}, F) \to (\mathbf{G}_1, F_1)$ be an epimorphism such that $\ker(\varphi)$ is central and let $\varphi^* \colon \mathbf{G}_1^* \to \mathbf{G}^*$ be a dual map. Let $s_1 \in \mathbf{G}_1^*$ such that $\varphi^*(s_1) = s$. Then $A_{\mathbf{G}_1}(s_1)^{F_1} = 1$.*

Proof. This follows from the discussion after [3, 2.A], applied to φ^*. □

Recall now that we let $\bar{\iota}$ be the specific regular embedding from Definition 8.1. Recall that $\phi : \mathbf{G} \to \mathbf{G}_1$ is an isotypy if the following hold: $\mathbf{G}$ and $\mathbf{G}_1$ are connected reductive, $[\phi(\mathbf{G}), \phi(\mathbf{G})] = [\mathbf{G}_1, \mathbf{G}_1]$, and $\ker(\phi) \leqslant \mathrm{Z}(\mathbf{G})$. The regular embedding $\bar{\iota}$ has the following lifting property with respect to isotypies.

Lemma 8.6. *Suppose $\phi : \mathbf{G} \to \mathbf{G}_1$ is an isotypy and let $\bar{\iota}$ be the regular embedding from Definition 8.1. Then there exists a regular embedding $\iota_1 : \mathbf{G}_1 \to \widetilde{\mathbf{G}}_1$ and an isotypy $\widetilde{\phi} : \widetilde{\mathbf{G}} \to \widetilde{\mathbf{G}}_1$ such that the following diagram commutes*

$$\begin{array}{ccc} \mathbf{G} & \xrightarrow{\phi} & \mathbf{G}_1 \\ \Big\downarrow{\scriptstyle \bar{\iota}} & & \Big\downarrow{\scriptstyle \iota_1} \\ \widetilde{\mathbf{G}} & \xrightarrow{\widetilde{\phi}} & \widetilde{\mathbf{G}}_1 \end{array}$$

Moreover, $\ker(\widetilde{\phi}) = \bar{\iota}(\ker(\phi))$ and if ϕ is surjective, then so is $\widetilde{\phi}$.

Proof. We have $\mathbf{T}_1 = \phi(\mathbf{T}) \cdot \mathrm{Z}(\mathbf{G}_1)^{\circ}$ is a maximal torus of $\mathbf{G}_1$. If we take $\widetilde{\mathbf{G}}_1 = (\mathbf{G}_1 \times \mathbf{T}_1)/\Delta_1$ and $\iota_1 : \mathbf{G}_1 \hookrightarrow \widetilde{\mathbf{G}}_1$ as above then we have a homomorphism $\pi : \mathbf{G} \times \mathbf{T} \to \widetilde{\mathbf{G}}_1$ defined by

$$\pi((g,t)) = (\phi(g), \phi(t))\Delta_1.$$

It is clear that $\iota_1(\phi(\mathbf{G}_1)) \leqslant \pi(\mathbf{G} \times \mathbf{T})$ contains the image of $\mathbf{G}_1$ in $\widetilde{\mathbf{G}}_1$.

We have $(g,t) \in \ker(\pi)$ if and only if $\phi(g) = \phi(t^{-1}) \in Z(\mathbf{G}_1)$. As $\ker(\phi) \leqslant Z(\mathbf{G})$ and $\mathbf{G}$ is reductive we have the assumptions of Lemma 6.3 hold so $g, t \in Z(\mathbf{G})$. Clearly $gt \in \ker(\phi)$ so

$$\ker(\pi) = \{(zt^{-1}, t) \mid z \in \ker(\phi) \text{ and } t \in Z(\mathbf{G}_1)\} = \bar{\iota}(\ker(\phi)) \cdot \Delta.$$

As this contains Δ we have π factors through a homomorphism $\widetilde{\phi} : \widetilde{\mathbf{G}} \to \widetilde{\mathbf{G}}_1$. The last two statements are clear. □

Hypothesis 8.7. Let $\bar{\iota} \colon \mathbf{G} \to \widetilde{\mathbf{G}}$ be the regular embedding constructed in Definition 8.1. Assume that $s \in G^*$ with $A_{\mathbf{G}}(s)^F = 1$. Assume we are in one of the following situations:

- We have fixed an epimorphism $\varphi \colon \mathbf{G} \to \mathbf{G}_1$ with central kernel, so that we obtain a regular embedding $\iota_1 \colon \mathbf{G}_1 \to \widetilde{\mathbf{G}}_1$ and an epimorphism $\widetilde{\varphi} \colon \widetilde{\mathbf{G}} \to \widetilde{\mathbf{G}}_1$, by Lemma 8.6.
- We have fixed an epimorphism $\widetilde{\varphi} \colon \widetilde{\mathbf{G}} \to \widetilde{\mathbf{G}}_1$ with central kernel, so we obtain an epimorphism $\varphi \colon \mathbf{G} \to \mathbf{G}_1$ with central kernel and a group $\mathbf{G}_1$ with $\iota_1 \colon \mathbf{G}_1 \to \widetilde{\mathbf{G}}_1$ a regular embedding by Lemma 8.4.

In either case, further let $s_1 \in \mathbf{G}_1^*$ be such that $\varphi^*(s_1) = s$, so that $A_{\mathbf{G}_1}(s_1)^F = 1$ by Lemma 8.5.

Note that in either situation of Hypothesis 8.7, we obtain commutative diagrams:

$$\begin{array}{ccc} \widetilde{\mathbf{G}} & \xrightarrow{\widetilde{\varphi}} & \widetilde{\mathbf{G}}_1 \\ \uparrow{\scriptstyle \bar{\iota}} & & \uparrow{\scriptstyle \iota_1} \\ \mathbf{G} & \xrightarrow{\varphi} & \mathbf{G}_1. \end{array} \quad \text{and} \quad \begin{array}{ccc} \widetilde{\mathbf{G}}^* & \xleftarrow{\widetilde{\varphi}^*} & \widetilde{\mathbf{G}}_1^* \\ \downarrow{\scriptstyle \bar{\iota}^*} & & \downarrow{\scriptstyle \iota_1^*} \\ \mathbf{G}^* & \xleftarrow{\varphi^*} & \mathbf{G}_1^*. \end{array}$$

.

We next consider the maps ${}^\top\varphi$ and ${}^\top\widetilde{\varphi}$.

Lemma 8.8. *Assume Hypothesis 8.7. Write $G_1 := \mathbf{G}_1^F$ and $\widetilde{G}_1 := \widetilde{\mathbf{G}}_1^F$. Then ${}^\top\varphi$ induces a map $\mathcal{E}(G_1, s_1) \to \mathcal{E}(G, s)$ and ${}^\top\widetilde{\varphi}$ induces a map $\mathcal{E}(\widetilde{G}_1, \widetilde{s}_1) \to \mathcal{E}(\widetilde{G}, \widetilde{s})$, where $\bar{\iota}^*(\widetilde{s}) = s$ and $\iota_1^*(\widetilde{s}_1) = s_1$.*

Proof. Write $K := (\ker\varphi)^F$. Note that $\varphi(G) = \varphi(\mathbf{G}^F) \cong G/K$ may be a proper subgroup of G_1. Here given $\chi_1 \in \mathcal{E}(G_1, s_1)$, we have ${}^\top\varphi(\chi_1)$, defined by ${}^\top\varphi(\chi_1)(g) = \chi_1(\varphi(g))$ gives a character of G. Note that ${}^\top\varphi(\chi_1)$ is the inflation to G of $\mathrm{Res}^{G_1}_{\varphi(G)}(\chi_1)$. We claim that this restriction is irreducible. Let χ_0 be a constituent of $\mathrm{Res}^{G_1}_{\varphi(G)}(\chi_1)$. Since $\varphi(G)$ contains $O^{p'}(G_1) = O^{p'}(\varphi(G)) = \varphi(O^{p'}(G))$, restrictions from G_1 to $O^{p'}(G_1)$ are multiplicity free by Theorem 6.7. As $G_1/\varphi(G)$ is abelian, we see it suffices to know that $(G_1)_{\chi_0} = G_1$.

Let $\chi \in \mathcal{E}(G, s)$ be the inflation of χ_0 to G. Since $A_{\mathbf{G}}(s)^F = 1$, we have χ extends to $\widetilde{G}$, so $\widetilde{G}_\chi = \widetilde{G}$. Then by Theorem 6.7 and Proposition 6.8, we also have χ extends to $\hat{G}$. Thus χ_0 extends to $\hat{G}/K \cong \varphi(\hat{G}) = \hat{G}_1$, which contains G_1 (see Lemma 6.4). This forces $(G_1)_{\chi_0} = G_1$, as desired.

Now, by Proposition 7.5, we have ${}^\top\varphi(\chi_1)$ further lies in $\mathcal{E}(G, s)$ since $s = \varphi^*(s_1)$. A similar argument shows that ${}^\top\widetilde{\varphi}$ gives a map $\mathcal{E}(\widetilde{G}_1, \widetilde{s}_1) \to \mathcal{E}(\widetilde{G}, \widetilde{s})$. □

With this, we have:

Lemma 8.9. *Assume Hypothesis 8.7. Write $G_1 := \mathbf{G}_1^F$ and $\widetilde{G}_1 := \widetilde{\mathbf{G}}_1^F$. Then the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{E}(G, s) & \xrightarrow{\Upsilon_s} & \mathcal{E}(\widetilde{G}, \widetilde{s}) \\ \uparrow{\scriptstyle {}^\top\varphi} & & \uparrow{\scriptstyle {}^\top\widetilde{\varphi}} \\ \mathcal{E}(G_1, s_1) & \xrightarrow{\Upsilon_{s_1}} & \mathcal{E}(\widetilde{G}_1, \widetilde{s}_1), \end{array}$$

where Υ_s is the inverse of the restriction map $\mathrm{Res}^{\widetilde{G}}_G \colon \mathcal{E}(\widetilde{G}, \widetilde{s}) \to \mathcal{E}(G, s)$ and similar for Υ_{s_1}.

Proof. From Lemma 8.8, we see that the maps ${}^\top\varphi$ and ${}^\top\widetilde{\varphi}$ in the diagram are well-defined. Further, recalling that $A_{\mathbf{G}_1}(s_1)^{F_1} = 1$ by Lemma 8.5, we have Υ_{s_1} is well-defined, by Lemma 3.2. We now show that the diagram commutes.

Let $\chi_1 \in \mathcal{E}(G_1, s_1)$ and let $\widetilde{\chi}_1 \in \mathcal{E}(\widetilde{G}_1, \widetilde{s}_1)$ such that $\mathrm{Res}^{\widetilde{G}_1}_{G_1} \widetilde{\chi}_1 = \chi_1$. (That is, $\widetilde{\chi}_1 = \Upsilon_{s_1}(\chi_1)$.) We claim that ${}^\top\varphi(\chi_1)$ is a constituent of $\mathrm{Res}^{\widetilde{G}}_{G}\left({}^\top\widetilde{\varphi}(\widetilde{\chi}_1)\right)$, which will imply that $\Upsilon_s \circ {}^\top\varphi(\chi_1) = {}^\top\widetilde{\varphi} \circ \Upsilon_{s_1}(\chi_1)$.

We have

$$\langle {}^\top\varphi(\chi_1), \mathrm{Res}^{\widetilde{G}}_{G}\left({}^\top\widetilde{\varphi}(\widetilde{\chi}_1)\right)\rangle_G = \frac{1}{|G|}\sum_{g\in G}\chi_1(\varphi(g))\overline{\widetilde{\chi}_1(\widetilde{\varphi}(\widetilde{\iota}(g)))}$$

$$= \frac{1}{|G|}|(\ker\varphi)^F| \sum_{x\in G_1}\chi_1(x)\overline{\widetilde{\chi}_1(\iota_1(x)))} = \frac{1}{|G_1|}\sum_{x\in G_1}\chi_1(x)\overline{\widetilde{\chi}_1(\iota_1(x)))}.$$

But this is just $\langle \chi_1, \mathrm{Res}^{\widetilde{G}_1}_{G_1}\widetilde{\chi}_1\rangle_{G_1}$, completing the proof. □

Lemma 8.10. *Assume Hypothesis 8.7. Then the following diagram commutes:*

$$\begin{array}{ccc}
\mathcal{E}(C_{\widetilde{G}^*}(\widetilde{s}), 1) & \xrightarrow{{}^\top\widetilde{\varphi}^*} & \mathcal{E}(C_{\widetilde{G}_1^*}(\widetilde{s}_1), 1) \\
\big\downarrow I_s & & \big\downarrow I_{s_1} \\
\mathcal{E}(C_{G^*}(s), 1) & \xrightarrow{{}^\top\varphi^*} & \mathcal{E}(C_{G_1^*}(s_1), 1),
\end{array}$$

where we define $I_s := ({}^\top\widetilde{\iota}^*)^{-1}$ and $I_{s_1} := ({}^\top\iota_1^*)^{-1}$.

Proof. Note that the maps ${}^\top\varphi^*$ and ${}^\top\widetilde{\varphi}^*$ in the diagram make sense, as $A(s)^F$ and $A(s_1)^F$ are both trivial.

Let $\widetilde{\psi} \in \mathcal{E}(C_{\widetilde{G}^*}(\widetilde{s}), 1)$ and let $\psi = I_s(\widetilde{\psi})$, so that $\widetilde{\psi} = \psi \circ \widetilde{\iota}^*$. Since $\widetilde{\iota}^* \circ \widetilde{\varphi}^* = \varphi^* \circ \iota_1^*$, we see for $g \in C_{\widetilde{G}_1^*}(\widetilde{s}_1)$, we have ${}^\top\varphi^*(\psi)(\iota_1^*(g)) = \psi(\varphi^*(\iota_1^*(g))) = \psi(\widetilde{\iota}^*(\widetilde{\varphi}^*(g))) = \widetilde{\psi}(\widetilde{\varphi}^*(g)) = {}^\top\widetilde{\varphi}^*(\widetilde{\psi})(g)$. This means ${}^\top\varphi^*(\psi) \circ \iota_1^* = {}^\top\widetilde{\varphi}^*(\widetilde{\psi})$, and hence $I_{s_1} \circ {}^\top\widetilde{\varphi}^* = {}^\top\varphi^* \circ I_s$. □

Corollary 8.11. *Assume Hypothesis 8.7. Let $f_s\colon \mathcal{E}(G,s) \to \mathcal{E}(C_{G^*}(s), 1)$ and $f_{\widetilde{s}}\colon \mathcal{E}(\widetilde{G}, \widetilde{s}) \to \mathcal{E}(C_{\widetilde{G}^*}(\widetilde{s}), 1)$ be bijections constructed such that the following diagram is commutative:*

$$\begin{array}{ccc}
\mathcal{E}(\widetilde{G}, \widetilde{s}) & \xrightarrow{f_{\widetilde{s}}} & \mathcal{E}(C_{\widetilde{G}^*}(\widetilde{s}), 1) \\
\big\downarrow \mathrm{Res}^{\widetilde{G}}_{G} & & \big\uparrow {}^\top\widetilde{\iota}^* \\
\mathcal{E}(G, s) & \xrightarrow{f_s} & \mathcal{E}(C_{G^*}(s), 1)
\end{array}$$

Let f_{s_1} and $f_{\widetilde{s}_1}$ be analogous. Then the diagram

$$\begin{array}{ccc}
\mathcal{E}(G, s) & \xrightarrow{f_s} & \mathcal{E}(C_{G^*}(s), 1) \\
\big\uparrow {}^\top\varphi & & \big\downarrow {}^\top\varphi^* \\
\mathcal{E}(G_1, s_1) & \xrightarrow{f_{s_1}} & \mathcal{E}(C_{G_1^*}(s_1), 1),
\end{array}
\quad \textit{commutes if and only if} \quad
\begin{array}{ccc}
\mathcal{E}(\widetilde{G}, \widetilde{s}) & \xrightarrow{f_{\widetilde{s}}} & \mathcal{E}(C_{\widetilde{G}^*}(\widetilde{s}), 1) \\
\big\uparrow {}^\top\widetilde{\varphi} & & \big\downarrow {}^\top\widetilde{\varphi}^* \\
\mathcal{E}(\widetilde{G}_1, \widetilde{s}_1) & \xrightarrow{f_{\widetilde{s}_1}} & \mathcal{E}(C_{\widetilde{G}_1^*}(\widetilde{s}_1), 1)
\end{array}$$

does.

Proof. Consider the 3-D diagram whose top and bottom diagrams are those discussed regarding $f_s, f_{\widetilde{s}}$ and $f_{s_1}, f_{\widetilde{s}_1}$; whose left and right diagrams are those discussed in Lemmas 8.9 and 8.10, and whose back and front diagrams are those being considered here. Since the top and bottom diagrams commute by assumption (which makes sense by the fact that $A(s)^F = 1 = A(s_1)^{F_1}$) and the left and right diagrams commute by Lemmas 8.9 and 8.10, we see that the back commutes if and only if the front commutes. □

We now immediately obtain:

Corollary 8.12. *Keep the hypotheses of Proposition 8.2. Then Property (6) of Condition 5.1 holds for $f_{s,\overline{\iota}}$ if and only if Property (6) of Theorem 2.1 holds for $f_{\widetilde{s}}$.*

We have now completed the proof of Proposition 8.2, by combining Corollary 8.12 with Lemma 8.3.

9. Uniqueness of Jordan Decomposition in Case $A(s)^F = 1$

We now complete the proof of Theorem 1.1, with the following:

Theorem 9.1. *For each $(\mathbf{G}, F)$ a finite reductive group defined over $\mathbb{F}_q$, let $\mathcal{D} = ((\mathbf{G}, F), (\mathbf{G}^*, F^*), \mathscr{T})$ be a fixed rational duality. Then there is a unique set of bijections*

$$J_s^{\mathbf{G}} : \mathcal{E}(G, s) \longrightarrow \mathcal{E}(\mathrm{C}_{G^*}(s), 1),$$

indexed by $s \in \mathcal{S}_{\mathcal{D}}$, such that properties (1)-(7) of Condition 5.1 hold. Moreover, this bijection satisfies

(a) For the regular embedding $\overline{\iota}\colon \mathbf{G} \hookrightarrow \widetilde{\mathbf{G}}$ as in Definition 8.1 and any $\widetilde{s} \in \widetilde{G}^*$ such that $\overline{\iota}^*(\widetilde{s}) = s$, the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{E}(\widetilde{G}, s) & \xrightarrow{J_{\widetilde{s}}^{\widetilde{\mathbf{G}}}} & \mathcal{E}(\mathrm{C}_{\widetilde{G}^*}(s), 1) \\
\big\downarrow \mathrm{Res}_G^{\widetilde{G}} & & \big\uparrow {}^{\top}\overline{\iota}^* \\
\mathcal{E}(G, s) & \xrightarrow{J_s^{\mathbf{G}}} & \mathcal{E}(\mathrm{C}_{G^*}(s), 1)
\end{array}$$

where $J_{\widetilde{s}}^{\widetilde{\mathbf{G}}}$ is the unique map for $\widetilde{G}$ satisfying Theorem 2.1.

(b) Further, the collection $\{J_s^{\mathbf{G}} \mid A(s)^F = 1\}$ is $\mathcal{G}$-equivariant.

Proof. We take the regular embedding $\overline{\iota} : \mathbf{G} \to \widetilde{\mathbf{G}}$ as in Definition 8.1, with semisimple element $\tilde{s} \in \widetilde{G}^*$ such that $\overline{\iota}^*(\tilde{s}) = s$, and $J_{\tilde{s}}^{\widetilde{\mathbf{G}}}$ the uniquely defined Jordan decomposition map in Theorem 2.1. Then by Propositions 5.2, 5.6, and 8.2, the map $J_s^{\mathbf{G}}$ defined by

$$J_s^{\mathbf{G}} = ({}^{\top}\overline{\iota}^*)^{-1} \circ J_{\tilde{s}}^{\widetilde{\mathbf{G}}} \circ (\mathrm{Res}_G^{\widetilde{G}})^{-1}$$

satisfies properties (1)-(7) of Condition 5.1, giving the existence.

If $f_s : \mathcal{E}(G, s) \longrightarrow \mathcal{E}(\mathrm{C}_{G^*}(s), 1)$ is any other bijection satisfying these properties, then again from Propositions 5.2, 5.6, and 8.2, the map

$$f_{\tilde{s}} := {}^{\top}\overline{\iota}^* \circ f_s \circ \mathrm{Res}_G^{\widetilde{G}} : \mathcal{E}(\widetilde{G}, \tilde{s}) \to \mathcal{E}(\mathrm{C}_{G^*}(s), 1)$$

can be constructed for any $\widetilde{s}$ such that $\overline{\iota}^*(\widetilde{s}) = s$ and the collection of such maps satisfies properties (1)-(7) of Theorem 2.1. It follows from this result that $f_{\tilde{s}} = J_{\tilde{s}}^{\widetilde{\mathbf{G}}}$, and so we have

$$f_s = ({}^{\top}\overline{\iota}^*)^{-1} \circ J_{\tilde{s}}^{\widetilde{\mathbf{G}}} \circ (\mathrm{Res}_G^{\widetilde{G}})^{-1} = J_s^{\mathbf{G}},$$

giving uniqueness. This together with Lemmas 4.1 and 4.2 yield (a) and (b). □

References

[1] M. F. Atiyah and I. G. Macdonald, Introduction to commutative algebra. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[2] C. Bonnafé, Mackey formula in type A, *Proc. London Math. Soc.*, **80** (2000), 545–574.

[3] C. Bonnafé, Quasi-isolated elements in reductive groups, *Comm. Algebra*, **33** (2005), no. 7, 2315–2337.

[4] M. Cabanes and M. Enguehard, Representation theory of finite reductive groups. New Mathematical Monographs, 1, Cambridge University Press, Cambridge, 2004.

[5] R. Carter, Finite groups of Lie type. Conjugacy classes and complex characters. Pure and Applied Mathematics (New York), John Wiley & Sons, Inc., New York, 1985.

[6] F. Digne and J. Michel, On Lusztig's parametrization of characters of finite groups of Lie type, *Astérisque* **No. 181-182** (1990), 6, 113–156.

[7] F. Digne and J. Michel, Representations of finite groups of Lie type. London Mathematical Society Student Texts, 95, Cambridge University Press, Cambridge, 2020.

[8] M. Geck, Character values, Schur indices, and character sheaves, *Represent. Theory* **7** (2003), 19–55.

[9] M. Geck and G. Malle, The character theory of finite groups of Lie type: A guided tour. Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2020.

[10] I. M. Isaacs, Character theory of finite groups. AMS Chelsea Publishing, Providence, Rhode Island, 2006.

[11] C. Li, Restrictions of irreducible characters of finite groups of Lie type to derived subgroups and regular semisimple elements. *Comm. Algebra* **51** (2023), no. 3, 983–990.

[12] G. Lusztig, Coxeter orbits and eigenspaces of Frobenius, *Invent. Math.* **38** (1976), 101–159.

[13] G. Lusztig, On a theorem of Benson and Curtis. *J. Algebra* **71** (1981), no. 2, 490–498.

[14] G. Lusztig, Characters of reductive groups over a finite field. Annals of Mathematics Studies, 107, Princeton University Press, Princeton, NJ, 1984.

[15] G. Lusztig, On the representations of reductive groups with disconnected centre. Orbites unipotentes et représentations, I, *Astérisque* **No. 168** (1988), 10, 157–166.

[16] G. Lusztig, Irreducible representations of finite spin groups. *Represent. Theory* **12** (2008), 1–36.

[17] G. Lusztig and N. Spaltenstein, On the generalized Springer correspondence for classical groups.Algebraic groups and related topics (Kyoto/Nagoya, 1983), 289–316. Adv. Stud. Pure Math., 6, North-Holland Publishing Co., Amsterdam, 1985.

[18] G. Navarro, B. Späth, and C. Vallejo. A reduction theorem for the Galois–McKay conjecture. *Trans. Amer. Math. Soc.*, **373** (2020), no. 9, 6157–6183.

[19] L. Ruhstorfer. Derived equivalences and equivariant Jordan decomposition, *Represent. Theory* **26** (2022), 542–584.

[20] L. Ruhstorfer and A. A. Schaeffer Fry. The inductive McKay–Navarro conditions for the prime 2 and some groups of Lie type, *Proc. Amer. Math. Soc. Ser. B* **9** (2022), 204–220.

[21] L. Ruhstorfer and A. A. Schaeffer Fry. Navarro's Galois–McKay conjecture for the prime 2, Preprint arXiv: 2211.14237, 2022.

[22] A. A. Schaeffer Fry, Galois automorphisms on Harish-Chandra series and Navarro's self-normalizing Sylow 2-subgroup conjecture, *Trans. Amer. Math. Soc.* **372** (2019), no. 1, 457–483.

[23] A. A. Schaeffer Fry and J. Taylor, On self-normalising Sylow 2-subgroups in type A, *J. Lie Theory* **28** (2018), no. 1, 139–168.

[24] A. A. Schaeffer Fry and J. Taylor, Galois automorphisms and classical groups, *Transform. Groups* **28** (2023), 439–486.

[25] A. A. Schaeffer Fry and C. R. Vinroot, Fields of character values for finite special unitary groups, *Pacific J. Math.* **300** (2019), No. 2, 473–489.

[26] B. Srinivasan and C. R. Vinroot, Jordan decomposition and real-valued characters of finite reductive groups with connected center, *Bull. Lond. Math. Soc.* **47** (2015), no. 3, 427–435.

[27] B. Srinivasan and C. R. Vinroot, Galois group action on Jordan decomposition of finite reductive groups with connected center, *J. Algebra* **558** (2020), 708–727.

[28] R. Steinberg, Endomorphisms of linear algebraic groups. Memoirs of the American Mathematical Society, No. 80 American Mathematical Society, Providence, R.I. 1968.

[29] J. Taylor, Action of automorphisms on irreducible characters of symplectic groups, *J. Algebra* **505** (2018), 211–246.

[30] J. Taylor, The structure of root data and smooth regular embeddings of reductive groups, *Proc. Edinb. Math. Soc.* **62** (2019), no.2, 523–552.

[31] J. Taylor, Harish-Chandra cuspidal pairs, *Indag. Math. (N.S.)* **33** (2022), no. 1, 24–38.

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