BUDGET VARIANCE ANALYSIS OF n-VARIABLE PRODUCTS WITH ZERO OR n RESPONSIBILITY CENTERS

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ABSTRACT

This article aims to propose a new set of budget variance analysis models, specifically, those for \( n \)-variable products with zero or \( n \) responsibility centers. We discuss the benefits of these new models over the biased, textbook model which is commonly used. We justify the mathematics of the proposed models with a formal proof. In doing so, we demonstrate that the associated differences of products can be expressed as a function of averages and differences of individual values. The models can be used in all Business disciplines currently using variance analysis, such as Accounting, Economics, Finance, Operations Management and Marketing.

INTRODUCTION

Variance analysis is a well-known and widely-used accounting tool for tracking business performance. Unfortunately, the most common model (used for the product of two variables) is biased in favor of one of its component variables and can yield misleading results. We show this with a demonstration using two-variable (unit price and quantity sold) revenue, including a numerical example. We propose a new set of unbiased models that incorporate the concept of a responsibility center (i.e., accountable manager, decision maker, department, etc.), supporting them with a mathematical proof. For an \( n \)-variable product, there can be \( 0, 1, 2, \ldots, n \) responsibility centers and the models differ based on both the number and combination of responsibility centers. We focus on a proposed set of models for an \( n \)-variable product with \( n \) responsibility centers. For such models, the variance of each component variable can be expressed as simply as a function of averages and differences of individual values.

Consider first, two-variable revenue (i.e., revenue is calculated as the product of two variables, unit price and quantity sold). Let \( p \) be the unit price of a product and \( q \) be the quantity sold of that product.
Furthermore, let
- \( p_b \) be the budgeted unit price,
- \( p_a \) be the actual unit price,
- \( q_b \) be the budgeted quantity sold, and
- \( q_a \) be the actual quantity sold.

Define the difference of unit price and the difference of quantity sold as
\[
\Delta p = p_a - p_b \quad \text{and} \quad \Delta q = q_a - q_b.
\]

Define the average unit price and the average quantity sold as
\[
\bar{p} = \frac{p_a + p_b}{2} \quad \text{and} \quad \bar{q} = \frac{q_a + q_b}{2}.
\]

The expression
\[
p_a q_a - p_b q_b
\]
is the difference of actual and budgeted revenues. Five key observations concerning this formula are given next. First, the most commonly-found variance analysis model for this expression is captured by
\[
p_a q_a - p_b q_b = \Delta p q_a - \Delta q p_b \quad (1)
\]
where the first term on the right-hand side of the equation is the portion of the difference in revenue corresponding to the difference in unit price (i.e., price variance) and the second term is the portion of the difference in revenue corresponding to the difference of quantity sold (i.e., quantity variance). Even though the model is ubiquitous in textbooks and practice, criticisms include “... the textbook example is solved by arbitrarily adding the joint variance to the price variance. There is no theoretical justification for so doing.” (Kloock, J. & Schiller, U., 1997), and “... the conventional two-variance analysis (price and quantity) inflates variances in three of the four possible economic situations.” (Mitchell, T. & Thomas, M., 2005). Such comments lead to an obvious question, “Why is Equation 1 used, as opposed to
\[
p_a q_a - p_b q_b = \Delta p q_b - \Delta q p_a? \quad (2)
\]
Second, we build upon Sorochuk et al. (2023) and incorporate the concept of responsibility centers. We propose 1) Model 1 is appropriate for a firm with a decision maker responsible
for unit price, but there is no decision maker responsible for quantity sold (e.g., a pricing manager sets the price for its product, but there is no active sales effort beyond making the product available for sale), 2) Model 2 is appropriate for a firm with a decision maker responsible for quantity sold, but there is no decision maker responsible for setting unit price (e.g., a firm that has an active sales force responsible for selling a pure commodity at a spot price determined by the market), and 3) for a firm with decision makers accountable for unit price and quantity sold (e.g., a cartel that can both ration units sold in the marketplace and set the selling price) the difference of revenues can be partitioned as

\[ p_a q_a - p_b q_b = \Delta p\bar{q} + \Delta q\bar{p}. \]  

As with Models 1 and 2, the first term on the right-hand side of the equation is the price variance and the second term is the quantity variance. Third, we illustrate the application of a generalization of this formula to the case of more than two factors (see Result section). Fourth, the application of this formula is not limited to just variance analysis. It can also be applied to two-period horizontal analysis, for example, comparing the revenues from two time periods. An example is a four-factor planning model which appears in Marketing textbooks (Spiro et al., 2003). Finally, the models discussed in this article and the algorithm used to generate them are not limited to just accounting applications. The algorithm can be used to generate a solution to the well-known Bankruptcy Problem in game theory (Aumann and Maschler, 1985) with \( n \) creditors collectively having a sum of claims greater than the value of the bankrupt firm. Another obvious game-theoretic application is executive compensation. Consider \( n \) executives discussing \textit{ex ante} how to assign credit or blame should a revenue or spending variance occur. The models discussed in this article are neutral and can be agreed upon in advance to calculate unbiased variances after actual results are recognized.

We continue with a numerical example demonstrating the bias inherent to Models 1 and 2. Where applicable, the marketing department is responsible for setting unit price and the sales department is responsible for quantity sold. Shown in Table 1 are the three models discussed above. Model 1 corresponds to Equation 1 and is the proposed model for when
the marketing department is the only responsibility center. Model 2 corresponds to Equation 2 and is the proposed model for when the sales department is the only responsibility center. Model 3 corresponds to Equation 3 and is the proposed model for when there are two separate responsibility centers: the marketing department and sales department.

**TABLE 1**

Variance Models for Two-Variable Revenue

<table>
<thead>
<tr>
<th>Variance</th>
<th>Marketing</th>
<th>Sales</th>
<th>Marketing and Sales*</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
</tr>
<tr>
<td>Price</td>
<td>(\Delta pq_a)</td>
<td>(\Delta pq_b)</td>
<td>(\Delta p\bar{q})</td>
</tr>
<tr>
<td>Quantity</td>
<td>(\Delta qp_b)</td>
<td>(\Delta qp_a)</td>
<td>(\Delta q\bar{p})</td>
</tr>
<tr>
<td>Revenue</td>
<td>(p_aq_a - p_bq_b)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Also applicable for zero responsibility centers.

Table 2 shows the parameters for the two, mirror-image cases used in the example. The motivation for this setup is as follows. One might intuitively think that the price and quantity variances for a given case would be the mirror image of the respective variances of the other case, given the parameters are mirror images of each other. Showing otherwise would justify investigation. As seen in Table 3, both models demonstrate a bias in favor of one component variable. A discussion follows.

Model 1 is the standard textbook model and is being used by most firms. That said, we propose it for a firm that has a pricing responsibility center (Marketing), but no sales responsibility center. As an example, consider a firm that sells products on Amazon.com. The Marketing department sets the price, but there is no analogous responsibility center playing an active role in promoting sales. The bias exists in favor of the only responsibility center (Marketing) that exists. There is $200 worth of credit in Case 1, but only $100 worth
of blame in Case 2.

**TABLE 2**

Parameters for Two-Variable Revenue Example

<table>
<thead>
<tr>
<th>Variable</th>
<th>Case 1</th>
<th></th>
<th>Case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Budgeted</td>
<td>Actual</td>
<td>Budgeted</td>
</tr>
<tr>
<td>Unit Price, $p$</td>
<td>$10</td>
<td>$11</td>
<td>$11</td>
</tr>
<tr>
<td>Quantity Sold, $q$</td>
<td>100</td>
<td>200</td>
<td>200</td>
</tr>
</tbody>
</table>

**TABLE 3**

Results of Two-Variable Revenue Example

<table>
<thead>
<tr>
<th>Variance</th>
<th>Case 1</th>
<th></th>
<th>Case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Responsibility Center(s)</td>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td>Price</td>
<td>Marketing</td>
<td>$200</td>
<td>$100</td>
</tr>
<tr>
<td>Quantity</td>
<td>Sales</td>
<td>$1000</td>
<td>$1100</td>
</tr>
<tr>
<td>Revenue</td>
<td>Revenue</td>
<td>$1200</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Variance</th>
<th>Case 2</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Responsibility Center(s)</td>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td>Price</td>
<td>Marketing</td>
<td>($100)</td>
<td>($200)</td>
</tr>
<tr>
<td>Quantity</td>
<td>Sales</td>
<td>($1100)</td>
<td>($1000)</td>
</tr>
<tr>
<td>Revenue</td>
<td>Revenue</td>
<td>($1200)</td>
<td></td>
</tr>
</tbody>
</table>

Model 2 is for a firm that has a sales responsibility center (Sales), but no pricing responsibility center. As an example, consider a firm that employs a sales department and sells a pure
commodity. The sales department is active and accountable, but the selling price is determined by an outside agency, such as a spot price on a world market. Similar to Model 1, bias exists in favor of the only responsibility center that exists. There is $1100 worth of credit in Case 1, but $1000 worth of blame in Case 2.

Model 3 is our proposed model for a firm that employs both a pricing responsibility center (Marketing) and a separate sales responsibility center (Sales). The results found using Model 3 demonstrate its unbiased nature. For both responsibility centers, the respective price and quantity variance for a given case are the same magnitude as those for the other case ($150 credit vs. $150 blame and $1050 credit vs. $1050 blame). However, if Model 1 is being used instead, it is apparent the impacts of the resulting bias can be significant. The Sales responsibility center is receiving a dearth of credit when things are good ($1000 vs. $1050), and a disproportionate amount of blame when things are bad ($1100 vs. $1050). Conversely, the Marketing responsibility center is receiving excess credit when things are good ($200 vs. $150), and a disproportionate amount of blame when things are bad ($100 vs. $150). This highlights the potential impact of using a biased model.

RESULT

The theorem below generalizes the example in the introduction to apply to more than just the $n = 2$ variables, unit price and quantity sold. The theorem considers $n$ variables that can each assume two values. The result indicates that the difference of the products can be written as a function of the averages and differences between the two values. Specific results for $n = 2, 3$ and 4 follow.

Theorem 1. Consider the variables $x_i$, for $i = 1, 2, \ldots, n$ that can each only assume the two values $x_{i,1}$ and $x_{i,2}$, for $i = 1, 2, \ldots, n$. Define the difference of $x_i$ as

$$\Delta x_i = x_{i,2} - x_{i,1}$$

and the average of $x_i$ as

$$\bar{x}_i = \frac{x_{i,2} + x_{i,1}}{2}$$
for \( i = 1, 2, ..., n \). The difference of the products can be expressed as

\[
\prod_{i=1}^{n} x_{i,2} - \prod_{i=1}^{n} x_{i,1} = \sum_{d=1,d \text{ odd}}^{n} \frac{1}{2^{d-1}} \sum_{S \subseteq [n], |S| = d} \prod_{j \in S, k \in S'} \Delta x_j \bar{x}_k,
\]

where \([n] = \{1, 2, ..., n\}\), \( S \) is any subset of \([n]\), and \( S' \) is the complement of \( S \). The terms in the implementation of the right-hand sides of Equation 4 for \( n = 2, 3, ..., 6 \) are given in Table 4.

As specific examples, we generate and present our proposed models for two-variable revenue (the product of unit price \( p \) and quantity sold \( q \)), three-variable direct materials spending (the product of unit cost \( c \), quantity sold \( q \) and usage \( u \)) and four-variable direct materials spending (the product of unit cost \( c \), quantity sold \( q \), usage \( u \) and exchange rate, \( x \)) using the results in the first three columns of Table 4. See Tables 5, 6 and 7. Note that for any term in Table 4 that includes more than one \( \Delta \) factor, the term is divided equally among the respective variances for each \( \Delta \) factor. For example, \( \Delta x_1 \Delta x_2 \Delta x_3 / 4 \) is divided equally among \( \Delta x_1 \) variance, \( \Delta x_2 \) variance and \( \Delta x_3 \) variance.

**DISCUSSION**

A formal proof of Theorem 1 is given in the appendix. We give the motivation associated with the proof here. The \( 1/2^{d-1} \) factor in the result is used to account for the 2 in the denominator of the averages. So temporarily writing \( x_{i,1} \) as \( b_i \) and \( x_{i,2} \) as \( a_i \), we need to show that

\[
\prod_{i=1}^{n} x_{i,2} - \prod_{i=1}^{n} x_{i,1}
\]

is a function of the averages and differences of the \( b_i \) and \( a_i \) values. The choice of the variables’ names \( a_i \) and \( b_i \) is consistent with their interpretation as *actual* and *budgeted* values. When \( n = 4 \), for example, we want to show that the difference of the products is
### TABLE 4
Terms on the Right-Hand Sides of Equation 4.

<table>
<thead>
<tr>
<th>$n = 2$</th>
<th>$n = 3$</th>
<th>$n = 4$</th>
<th>$n = 5$</th>
<th>$n = 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta x_1 \bar{x}_2$</td>
<td>$\Delta x_1 \bar{x}_2 \bar{x}_3$</td>
<td>$\Delta x_1 \bar{x}_2 \bar{x}_3 \bar{x}_4$</td>
<td>$\Delta x_1 \bar{x}_2 \bar{x}_3 \bar{x}_4 \bar{x}_5$</td>
<td>$\Delta x_1 \bar{x}_2 \bar{x}_3 \bar{x}_4 \bar{x}_5 \bar{x}_6$</td>
</tr>
<tr>
<td>$\bar{x}_1 \Delta x_2$</td>
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</tr>
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</tr>
<tr>
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</tr>
</tbody>
</table>
### TABLE 5

**Variance Analysis Model for Two-Variable Revenue**

<table>
<thead>
<tr>
<th>Variable</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unit Price, $p$</td>
<td>$\Delta p\bar{q}$</td>
</tr>
<tr>
<td>Quantity Sold, $q$</td>
<td>$\Delta q\bar{p}$</td>
</tr>
<tr>
<td>Revenue, $R(p, q)$</td>
<td>$p_a q_a - p_b q_b$</td>
</tr>
</tbody>
</table>

### TABLE 6

**Variance Analysis Model for Three-Variable Spending**

<table>
<thead>
<tr>
<th>Variable</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unit Cost, $c$</td>
<td>$\Delta c \left( \bar{q}\bar{u} + \frac{\Delta q\Delta u}{12} \right)$</td>
</tr>
<tr>
<td>Quantity Sold, $q$</td>
<td>$\Delta q \left( \bar{c}\bar{u} + \frac{\Delta c\Delta u}{12} \right)$</td>
</tr>
<tr>
<td>Usage, $u$</td>
<td>$\Delta u \left( \bar{c}\bar{q} + \frac{\Delta c\Delta q}{12} \right)$</td>
</tr>
<tr>
<td>Spending, $S(c, q, u)$</td>
<td>$c_a q_a u_a - c_b q_b u_b$</td>
</tr>
</tbody>
</table>

### TABLE 7

**Variance Analysis Model for Four-Variable Spending**

<table>
<thead>
<tr>
<th>Variable</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unit Cost, $c$</td>
<td>$\Delta c \left( \bar{q}\bar{u}\bar{x} + \frac{\Delta q\Delta u\Delta x + \bar{q}\Delta u\Delta x}{12} \right)$</td>
</tr>
<tr>
<td>Quantity Sold, $q$</td>
<td>$\Delta q \left( \bar{c}\bar{u}\bar{x} + \frac{\Delta c\Delta u\Delta x + \bar{c}\Delta u\Delta x}{12} \right)$</td>
</tr>
<tr>
<td>Usage, $u$</td>
<td>$\Delta u \left( \bar{c}\bar{q}\bar{x} + \frac{\Delta c\Delta q\Delta x + \bar{c}\Delta q\Delta x}{12} \right)$</td>
</tr>
<tr>
<td>Exchange Rate, $x$</td>
<td>$\Delta x \left( \bar{c}\bar{q}\bar{u} + \frac{\Delta c\Delta q\bar{u} + \bar{c}\Delta q\Delta u}{12} \right)$</td>
</tr>
<tr>
<td>Spending, $S(c, q, u, x)$</td>
<td>$c_a q_a u_a x_a - c_b q_b u_b x_b$</td>
</tr>
</tbody>
</table>
\[ a_1a_2a_3a_4 - b_1b_2b_3b_4 = \frac{1}{8} \left[ (a_1 - b_1)(a_2 + b_2)(a_3 + b_3)(a_4 + b_4) + \\
(a_1 + b_1)(a_2 - b_2)(a_3 + b_3)(a_4 - b_4) + \\
(a_1 + b_1)(a_2 + b_2)(a_3 - b_3)(a_4 + b_4) + \\
(a_1 - b_1)(a_2 + b_2)(a_3 + b_3)(a_4 - b_4) + \\
(a_1 - b_1)(a_2 - b_2)(a_3 - b_3)(a_4 + b_4) + \\
(a_1 - b_1)(a_2 + b_2)(a_3 - b_3)(a_4 - b_4) + \\
(a_1 + b_1)(a_2 - b_2)(a_3 - b_3)(a_4 + b_4) + \\
(a_1 + b_1)(a_2 - b_2)(a_3 - b_3)(a_4 - b_4) \right]. \]

The first four terms on the right-hand side of this equation have a single − and the next four terms on the right-hand side of this equation have three −’s in the terms. The key to the proof is to see that all of the terms on the right-hand side of this equation cancel except for the monomials \( a_1a_2a_3a_4 \) and \( b_1b_2b_3b_4 \).

**SUMMARY**

Using two-variable revenue as an example, we demonstrated how the commonly-used variance analysis model is biased in favor of one of its component variables. We discussed the opportunity for improvement by incorporating the concept of responsibility centers. We presented alternative models with the focus being on an \( n \)-variable product with zero or \( n \) responsibility centers. We supported the proposed models with a mathematical proof showing the difference of a product of \( n \) variables can be expressed as simply as a function of averages and differences. Specific revenue and spending models were presented.
REFERENCES


APPENDIX: PROOF OF THEOREM 1

For the purposes of this proof, let \( x_{i,1} = b_i \) and \( x_{i,2} = a_i \), for \( i = 1, 2, \ldots, n \). It is sufficient to show that

\[
\prod_{i=1}^{n} x_{i,2} - \prod_{i=1}^{n} x_{i,1} = \sum_{d=1, d \text{ odd}}^{n} \frac{1}{2^{d-1}} \sum_{S \subseteq [n], |S| = d} \prod_{j \in S, k \in S'} (a_j - b_j)(a_k + b_k),
\]  

(4)

so that the right-hand side of the equation will be written entirely in terms of sums and differences. The result is proven by showing that of all of the monomials resulting by multiplying out the right-hand side of this equation, for example, \( a_1 a_2 b_3 a_4 \ldots b_n \), all terms cancel except for \( \prod_{i=1}^{n} a_i \) and \( \prod_{i=1}^{n} b_i \). For the terms to cancel, there must be an equal number of positive and negative terms comprising the monomial. The number of odd-order subsets of \([n]\) is \( 2^{n-1} \). This can be seen by expanding \((1 - 1)^n\) by the binomial theorem:

\[
(1 - 1)^n \sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0
\]

Separating the negative and positive terms,

\[
\sum_{i=0, i \text{ odd}}^{n} \binom{n}{i} = \sum_{i=0, i \text{ even}}^{n} \binom{n}{i}
\]

Since

\[
\sum_{i=0}^{n} \binom{n}{i} = 2^n
\]

there are \( 2^{n-1} \) odd-ordered subsets and \( 2^{n-1} \) even-ordered subsets of \([n]\).

Now consider an arbitrary monomial resulting in multiplying out the terms on the right-hand side of Equation 4. To show that all terms except \( \prod_{i=1}^{n} a_i \) and \( \prod_{i=1}^{n} b_i \) cancel, consider the following cases involving a particularly arbitrary monomial other than \( \prod_{i=1}^{n} a_i \) or \( \prod_{i=1}^{n} b_i \).

**Case 1.** The number of times that the monomial is *negative* equals the number of odd-order subsets of \([n]\) containing an *odd* number of \( k \)-element index sets, which equals the product of the number of odd-order subsets of \( k \)-element sets and the number of even-order subsets of \((n - k)\)-element complement sets, which, by the multiplication rule, is
\[ 2^{k-1} \cdot 2^{n-k-1} = 2^{n-2}. \]

**Case 2.** The number of times that the monomial is *positive* equals the number of odd-order subsets of \([n]\) containing an *even* number of \(k\)-element index sets, which equals the product of the number of *even*-order subsets of \(k\)-element sets and the number of *odd*-order subsets of \((n - k)\)-element complement sets, which, by the multiplication rule, is

\[ 2^{k-1} \cdot 2^{n-k-1} = 2^{n-2}. \]

Since there are an equal number of positive and negative terms on any monomial term except \(\prod_{i=1}^{n} a_i\) and \(\prod_{i=1}^{n} b_i\), they must cancel. All of the \(2^{n-1}\) products involving the monomial \(a_1a_2 \ldots a_n\) are positive and all of the \(2^{n-1}\) products involving the monomial \(b_1b_2 \ldots b_n\) are negative, which proves the result.