BUDGET VARIANCE ANALYSIS OF *n*-VARIABLE PRODUCTS WITH ZERO OR *n* RESPONSIBILITY CENTERS

Craig M. Sorochuk, PhD
Stillman School of Business, Seton Hall University
400 South Orange Avenue
South Orange, NJ 07079
973-761-9000
craig.sorochuk@shu.edu

C. Ryan Vinroot, PhD
Department of Mathematics, William & Mary
Williamsburg, VA 23185-8795
757-221-5499
vinroot@math.wm.edu

Lawrence M. Leemis, PhD
Department of Mathematics, William & Mary
Williamsburg, VA 23185-8795
757-221-2034
lmleem@wm.edu

Keywords: Budget analysis, Combinations, Cost accounting, Financial analysis, Horizontal analysis, Management accounting, Margin drivers, Performance management, Responsibility centers.

BUDGET VARIANCE ANALYSIS OF *n*-VARIABLE PRODUCTS WITH ZERO OR *n* RESPONSIBILITY CENTERS

ABSTRACT

This article aims to propose a new set of budget variance analysis models, specifically, those for *n*-variable products with zero or *n* responsibility centers. We discuss the benefits of these new models over the biased, textbook model which is commonly used. We justify the mathematics of the proposed models with a formal proof. In doing so, we demonstrate that the associated differences of products can be expressed as a function of averages and differences of individual values. The models can be used in all Business disciplines currently using variance analysis, such as Accounting, Economics, Finance, Operations Management and Marketing.

INTRODUCTION

Variance analysis is a well-known and widely-used accounting tool for tracking business performance. Unfortunately, the most common model (used for the product of two variables) is biased in favor of one of its component variables and can yield misleading results. We show this with a demonstration using two-variable (unit price and quantity sold) revenue, including a numerical example. We propose a new set of unbiased models that incorporate the concept of a responsibility center (i.e., accountable manager, decision maker, department, etc.), supporting them with a mathematical proof. For an n-variable product, there can be $0, 1, 2, \ldots, n$ responsibility centers and the models differ based on both the number and combination of responsibility centers. We focus on a proposed set of models for an n-variable product with n responsibility centers. For such models, the variance of each component variable can be expressed as simply as a function of averages and differences of individual values.

Consider first, two-variable revenue (i.e., revenue is calculated as the product of two variables, unit price and quantity sold). Let p be the unit price of a product and q be the quantity sold of that product.

Furthermore, let

- p_b be the budgeted unit price,
- p_a be the actual unit price,
- q_b be the budgeted quantity sold, and
- q_a be the actual quantity sold.

Define the difference of unit price and the difference of quantity sold as

$$\Delta p = p_a - p_b \qquad \text{and} \qquad \Delta q = q_a - q_b.$$

Define the average unit price and the average quantity sold as

$$\bar{p} = \frac{p_a + p_b}{2} \qquad \text{and} \qquad \bar{q} = \frac{q_a + q_b}{2}.$$

The expression

$$p_a q_a - p_b q_b$$

is the difference of actual and budgeted revenues. Five key observations concerning this formula are given next. First, the most commonly-found variance analysis model for this expression is captured by

$$p_a q_a - p_b q_b = \Delta p q_a - \Delta q p_b \tag{1}$$

where the first term on the right-hand side of the equation is the portion of the difference in revenue corresponding to the difference in unit price (i.e., price variance) and the second term is the portion of the difference in revenue corresponding to the difference of quantity sold (i.e., quantity variance). Even though the model is ubiquitous in textbooks and practice, criticisms include "... the textbook example is solved by arbitrarily adding the joint variance to the price variance. There is no theoretical justification for so doing." (Kloock, J. & Schiller, U., 1997), and ". . . the conventional two-variance analysis (price and quantity) inflates variances in three of the four possible economic situations." (Mitchell, T. & Thomas, M., 2005). Such comments lead to an obvious question, "Why is Equation 1 used, as opposed to

$$p_a q_a - p_b q_b = \Delta p q_b - \Delta q p_a?" \tag{2}$$

Second, we build upon Sorochuk et al (2023) and incorporate the concept of responsibility centers. We propose 1) Model 1 is appropriate for a firm with a decision maker responsible for unit price, but there is no decision maker responsible for quantity sold (e.g., a pricing manager sets the price for its product, but there is no active sales effort beyond making the product available for sale), 2) Model 2 is appropriate for a firm with a decision maker responsible for quantity sold, but there is no decision maker responsible for setting unit price (e.g., a firm that has an active sales force responsible for selling a pure commodity at a spot price determined by the market), and 3) for a firm with decision makers accountable for unit price and quantity sold (e.g., a cartel that can both ration units sold in the marketplace and set the selling price) the difference of revenues can be partitioned as

$$p_a q_a - p_b q_b = \Delta p \bar{q} + \Delta q \bar{p}. \tag{3}$$

As with Models 1 and 2, the first term on the right-hand side of the equation is the price variance and the second term is the quantity variance. Third, we illustrate the application of a generalization of this formula to the case of more than two factors (see Result section). Fourth, the application of this formula is not limited to just variance analysis. It can also be applied to two-period horizontal analysis, for example, comparing the revenues from two time periods. An example is a four-factor planning model which appears in Marketing textbooks (Spiro et al., 2003). Finally, the models discussed in this article and the algorithm used to generate them are not limited to just accounting applications. The algorithm can be used to generate a solution to the well-known Bankruptcy Problem in game theory (Aumann and Maschler, 1985) with n creditors collectively having a sum of claims greater than the value of the bankrupt firm. Another obvious game-theoretic application is executive compensation. Consider n executives discussing *ex ante* how to assign credit or blame should a revenue or spending variance occur. The models discussed in this article are neutral and can be agreed upon in advance to calculate unbiased variances after actual results are recognized.

We continue with a numerical example demonstrating the bias inherent to Models 1 and 2. Where applicable, the marketing department is responsible for setting unit price and the sales department is responsible for quantity sold. Shown in Table 1 are the three models discussed above. Model 1 corresponds to Equation 1 and is the proposed model for when the marketing department is the only responsibility center. Model 2 corresponds to Equation 2 and is the proposed model for when the sales department is the only responsibility center. Model 3 corresponds to Equation 3 and is the proposed model for when there are two separate responsibility centers: the marketing department and sales department.

TABLE 1

Variance Models for Two-Variable Revenue

Variance	Responsibility Center(s)		
	Marketing (1)	Sales (2)	Marketing and Sales* (3)
Price	$\Delta p q_a$	$\Delta p q_b$	$\Delta p \bar{q}$
Quantity	$\Delta q p_b$	$\Delta q p_a$	$\Delta q \bar{p}$
Revenue	$p_a q_a - p_b q_b$		

*Also applicable for zero responsibility centers.

Table 2 shows the parameters for the two, mirror-image cases used in the example. The motivation for this setup is as follows. One might intuitively think that the price and quantity variances for a given case would be the mirror image of the respective variances of the other case, given the parameters are mirror images of each other. Showing otherwise would justify investigation. As seen in Table 3, both models demonstrate a bias in favor of one component variable. A discussion follows.

Model 1 is the standard textbook model and is being used by most firms. That said, we propose it for a firm that has a pricing responsibility center (Marketing), but no sales responsibility center. As an example, consider a firm that sells products on Amazon.com. The Marketing department sets the price, but there is no analogous responsibility center playing an active role in promoting sales. The bias exists in favor of the only responsibility center (Marketing) that exists. There is $200 worth of credit in Case 1, but only $100 worth of blame in Case 2.

TABLE 2

Parameters for Two-Variable Revenue Example

Variable	Case 1		Case 2	
	Budgeted	Actual	Budgeted	Actual
Unit Price, p	\$10	\$11	\$11	\$10
Quantity Sold, q	100	200	200	100

TABLE 3

Results of Two-Variable Revenue Example

Variance	Case 1		
	Responsibility Center(s)		
	Marketing (1)	Sales (2)	Marketing and Sales* (3)
Price	\$200	\$100	\$150
Quantity	\$1000	\$1100	\$1050
Revenue	\$1200		

Variance	Case 2		
	Responsibility Center(s)		
	Marketing (1)	Sales (2)	Marketing and Sales* (3)
Price	(\$100)	(\$200)	(\$150)
Quantity	(\$1100)	(\$1000)	(\$1050)
Revenue	(\$1200)		

Model 2 is for a firm that has a sales responsibility center (Sales), but no pricing responsibility center. As an example, consider a firm that employs a sales department and sells a pure commodity. The sales department is active and accountable, but the selling price is determined by an outside agency, such as a spot price on a world market. Similar to Model 1, bias exists in favor of the only responsibility center that exists. There is \$1100 worth of credit in Case 1, but \$1000 worth of blame in Case 2.

Model 3 is our proposed model for a firm that employs both a pricing responsibility center (Marketing) and a separate sales responsibility center (Sales). The results found using Model 3 demonstrate its unbiased nature. For both responsibility centers, the respective price and quantity variance for a given case are the same magnitude as those for the other case (\$150 credit vs. \$150 blame and \$1050 credit vs. \$1050 blame). However, if Model 1 is being used instead, it is apparent the impacts of the resulting bias can be significant. The Sales responsibility center is receiving a dearth of credit when things are good (\$1000 vs. \$1050), and a disproportionate amount of blame when things are bad (\$1100 vs. \$1050). Conversely, the Marketing responsibility center is receiving excess credit when things are good (\$200 vs. \$150), and a disproportionate amount of blame when things are bad (\$100 vs. \$150). This highlights the potential impact of using a biased model.

RESULT

The theorem below generalizes the example in the introduction to apply to more than just the $n = 2$ variables, unit price and quantity sold. The theorem considers n variables that can each assume two values. The result indicates that the difference of the products can be written as a function of the averages and differences between the two values. Specific results for $n = 2, 3$ and 4 follow.

Theorem 1. Consider the variables x_i, for $i = 1, 2, \dots, n$ that can each only assume the two values $x_{i,1}$ and $x_{i,2}$, for *i* = 1, 2, . . . , *n*. Define the difference of x_i as

$$\Delta x_i = x_{i,2} - x_{i,1}$$

and the average of x_i as

$$\bar{x}_i = \frac{x_{i,2} + x_{i,1}}{2}$$

for $i = 1, 2, \dots, n$. The difference of the products can be expressed as

$$\prod_{i=1}^{n} x_{i,2} - \prod_{i=1}^{n} x_{i,1} = \sum_{d=1, d\ odd}^{n} \frac{1}{2^{d-1}} \sum_{S\subseteq[n], |S|=d} \prod_{j\in S, k\in S'} \Delta x_j \bar{x}_k,$$

where $[n] = \{1, 2, \dots, n\}$, S is any subset of $[n]$, and S' is the complement of S. The terms in the implementation of the right-hand sides of Equation 4 for $n = 2, 3, \dots, 6$ are given in Table 4.

As specific examples, we generate and present our proposed models for two-variable revenue (the product of unit price p and quantity sold q), three-variable direct materials spending (the product of unit cost c, quantity sold q and usage u) and four-variable direct materials spending (the product of unit cost c, quantity sold q, usage u and exchange rate, x) using the results in the first three columns of Table 4. See Tables 5, 6 and 7. Note that for any term in Table 4 that includes more than one Δ factor, the term is divided equally among the respective variances for each Δ factor. For example, $\Delta x_1 \Delta x_2 \Delta x_3/4$ is divided equally among Δx_1 variance, Δx_2 variance and Δx_3 variance.

DISCUSSION

A formal proof of Theorem 1 is given in the appendix. We give the motivation associated with the proof here. The $1/2^{d-1}$ factor in the result is used to account for the 2 in the denominator of the averages. So temporarily writing $x_{i,1}$ as b_i and $x_{i,2}$ as a_i, we need to show that

$$\prod_{i=1}^{n} x_{i,2} - \prod_{i=1}^{n} x_{i,1}$$

is a function of the averages and differences of the b_i and a_i, values. The choice of the variables' names a_i and b_i is consistent with their interpretation as *actual* and *budgeted* values. When $n = 4$, for example, we want to show that the difference of the products is

TABLE 4

Terms on the Right-Hand Sides of Equation 4.

$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
$\Delta x_1\bar{x}_2$ $\bar{x}_1\Delta x_2$	$\Delta x_1\bar{x}_2\bar{x}_3$ $\bar{x}_1\Delta x_2\bar{x}_3$ $\bar{x}_1\bar{x}_2\Delta x_3$	$\Delta x_1\bar{x}_2\bar{x}_3\bar{x}_4$ $\bar{x}_1\Delta x_2\bar{x}_3\bar{x}_4$ $\bar{x}_1\bar{x}_2\Delta x_3\bar{x}_4$ $\bar{x}_1\bar{x}_2\bar{x}_3\Delta x_4$	$\Delta x_1\bar{x}_2\bar{x}_3\bar{x}_4\bar{x}_5$ $\bar{x}_1\Delta x_2\bar{x}_3\bar{x}_4\bar{x}_5$ $\bar{x}_1\bar{x}_2\Delta x_3\bar{x}_4\bar{x}_5$ $\bar{x}_1\bar{x}_2\bar{x}_3\Delta x_4\bar{x}_5$ $\bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4\Delta x_5$	$\Delta x_1\bar{x}_2\bar{x}_3\bar{x}_4\bar{x}_5\bar{x}_6$ $\bar{x}_1\Delta x_2\bar{x}_3\bar{x}_4\bar{x}_5\bar{x}_6$ $\bar{x}_1\bar{x}_2\Delta x_3\bar{x}_4\bar{x}_5\bar{x}_6$ $\bar{x}_1\bar{x}_2\bar{x}_3\Delta x_4\bar{x}_5\bar{x}_6$ $\bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4\Delta x_5\bar{x}_6$ $\bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4\bar{x}_5\Delta x_6$
	$\Delta x_1\Delta x_2\Delta x_3$	$\Delta x_1\Delta x_2\Delta x_3\bar{x}_4/4$ $\Delta x_1\Delta x_2\bar{x}_3\Delta x_4/4$ $\Delta x_1\bar{x}_2\Delta x_3\Delta x_4/4$ $\bar{x}_1\Delta x_2\Delta x_3\Delta x_4/4$	$\Delta x_1\Delta x_2\Delta x_3\bar{x}_4\bar{x}_5/4$ $\Delta x_1\Delta x_2\bar{x}_3\Delta x_4\bar{x}_5/4$ $\Delta x_1\Delta x_2\bar{x}_3\bar{x}_4\Delta x_5/4$ $\Delta x_1\bar{x}_2\Delta x_3\Delta x_4\bar{x}_5/4$ $\Delta x_1\bar{x}_2\Delta x_3\bar{x}_4\Delta x_5/4$ $\Delta x_1\bar{x}_2\bar{x}_3\Delta x_4\Delta x_5/4$ $\bar{x}_1\Delta x_2\Delta x_3\Delta x_4\bar{x}_5/4$ $\bar{x}_1\Delta x_2\Delta x_3\bar{x}_4\Delta x_5/4$ $\bar{x}_1\Delta x_2\bar{x}_3\Delta x_4\Delta x_5/4$ $\bar{x}_1\bar{x}_2\Delta x_3\Delta x_4\Delta x_5/4$	$\Delta x_1\Delta x_2\Delta x_3\bar{x}_4\bar{x}_5\bar{x}_6/4$ $\Delta x_1\Delta x_2\bar{x}_3\Delta x_4\bar{x}_5\bar{x}_6/4$ $\Delta x_1\Delta x_2\bar{x}_3\bar{x}_4\Delta x_5\bar{x}_6/4$ $\Delta x_1\Delta x_2\bar{x}_3\bar{x}_4\bar{x}_5\Delta x_6/4$ $\Delta x_1\bar{x}_2\Delta x_3\Delta x_4\bar{x}_5\bar{x}_6/4$ $\Delta x_1\bar{x}_2\Delta x_3\bar{x}_4\Delta x_5\bar{x}_6/4$ $\Delta x_1\bar{x}_2\Delta x_3\bar{x}_4\bar{x}_5\Delta x_6/4$ $\Delta x_1\bar{x}_2\bar{x}_3\Delta x_4\Delta x_5\bar{x}_6/4$ $\Delta x_1\bar{x}_2\bar{x}_3\Delta x_4\bar{x}_5\Delta x_6/4$ $\Delta x_1\bar{x}_2\bar{x}_3\bar{x}_4\Delta x_5\Delta x_6/4$ $\bar{x}_1\Delta x_2\Delta x_3\Delta x_4\bar{x}_5\bar{x}_6/4$ $\bar{x}_1\Delta x_2\Delta x_3\bar{x}_4\Delta x_5\bar{x}_6/4$ $\bar{x}_1\Delta x_2\Delta x_3\bar{x}_4\bar{x}_5\Delta x_6/4$ $\bar{x}_1\Delta x_2\bar{x}_3\Delta x_4\Delta x_5\bar{x}_6/4$ $\bar{x}_1\Delta x_2\bar{x}_3\Delta x_4\bar{x}_5\Delta x_6/4$ $\bar{x}_1\Delta x_2\bar{x}_3\bar{x}_4\Delta x_5\Delta x_6/4$ $\bar{x}_1\bar{x}_2\Delta x_3\Delta x_4\Delta x_5\bar{x}_6/4$ $\bar{x}_1\bar{x}_2\Delta x_3\Delta x_4\bar{x}_5\Delta x_6/4$ $\bar{x}_1\bar{x}_2\Delta x_3\bar{x}_4\Delta x_5\Delta x_6/4$ $\bar{x}_1\bar{x}_2\bar{x}_3\Delta x_4\Delta x_5\Delta x_6/4$
			$\Delta x_1\Delta x_2\Delta x_3\Delta x_4\Delta x_5/16$	$\Delta x_1\Delta x_2\Delta x_3\Delta x_4\Delta x_5\bar{x}_6/16$ $\Delta x_1\Delta x_2\Delta x_3\Delta x_4\bar{x}_5\Delta x_6/16$ $\Delta x_1\Delta x_2\Delta x_3\bar{x}_4\Delta x_5\Delta x_6/16$ $\Delta x_1\Delta x_2\bar{x}_3\Delta x_4\Delta x_5\Delta x_6/16$ $\Delta x_1\bar{x}_2\Delta x_3\Delta x_4\Delta x_5\Delta x_6/16$ $\bar{x}_1\Delta x_2\Delta x_3\Delta x_4\Delta x_5\Delta x_6/16$

TABLE 5

Variance Analysis Model for Two-Variable Revenue

Variable	Variance
Unit Price, p Quantity Sold, q	$\Delta p\bar{q}$ $\Delta q\bar{p}$
Revenue, $R(p,q)$	$p_a q_a - p_b q_b$

TABLE 6

Variance Analysis Model for Three-Variable Spending

Variable	Variance
Unit Cost, c Quantity Sold, q Usage, u	$\Delta c\left(\bar{q}\bar{u} + \frac{\Delta q \Delta u}{12}\right)$ $\Delta q\left(\bar{c}\bar{u} + \frac{\Delta c \Delta u}{12}\right)$ $\Delta u\left(\bar{c}\bar{q} + \frac{\Delta c \Delta q}{12}\right)$
Spending, $S(c,q,u)$	$c_a q_a u_a - c_b q_b u_b$

TABLE 7

Variance Analysis Model for Four-Variable Spending

Variable	Variance
Unit Cost, c Quantity Sold, q Usage, u Exchange Rate, x	$\Delta c\left(\bar{q}\bar{u}\bar{x} + \frac{\Delta q \Delta u \bar{x} + \Delta q \bar{u} \Delta x + \bar{q} \Delta u \Delta x}{12}\right)$ $\Delta q\left(\bar{c}\bar{u}\bar{x} + \frac{\Delta c \Delta u \bar{x} + \Delta c \bar{u} \Delta x + \bar{c} \Delta u \Delta x}{12}\right)$ $\Delta u\left(\bar{c}\bar{q}\bar{x} + \frac{\Delta c \Delta q \bar{x} + \Delta c \bar{q} \Delta x + \bar{c} \Delta q \Delta x}{12}\right)$ $\Delta x\left(\bar{c}\bar{q}\bar{u} + \frac{\Delta c \Delta q \bar{u} + \Delta c \bar{q} \Delta u + \bar{c} \Delta q \Delta u}{12}\right)$
Spending, $S(c,q,u,x)$	$c_a q_a u_a x_a - c_b q_b u_b x_b$

$$
\begin{aligned}
a_1a_2a_3a_4 - b_1b_2b_3b_4 = \frac{1}{8}[&(a_1 - b_1)(a_2 + b_2)(a_3 + b_3)(a_4 + b_4) + \\
&(a_1 + b_1)(a_2 - b_2)(a_3 + b_3)(a_4 + b_4) + \\
&(a_1 + b_1)(a_2 + b_2)(a_3 - b_3)(a_4 + b_4) + \\
&(a_1 + b_1)(a_2 + b_2)(a_3 + b_3)(a_4 - b_4) + \\
&(a_1 - b_1)(a_2 - b_2)(a_3 - b_3)(a_4 + b_4) + \\
&(a_1 - b_1)(a_2 - b_2)(a_3 + b_3)(a_4 - b_4) + \\
&(a_1 - b_1)(a_2 + b_2)(a_3 - b_3)(a_4 - b_4) + \\
&(a_1 + b_1)(a_2 - b_2)(a_3 - b_3)(a_4 - b_4)].
\end{aligned}
$$

The first four terms on the right-hand side of this equation have a single − and the next four terms on the right-hand side of this equation have three −'s in the terms. The key to the proof is to see that all of the terms on the right-hand side of this equation cancel except for the monomials $a_1a_2a_3a_4$ and $b_1b_2b_3b_4$.

SUMMARY

Using two-variable revenue as an example, we demonstrated how the commonly-used variance analysis model is biased in favor of one of its component variables. We discussed the opportunity for improvement by incorporating the concept of responsibility centers. We presented alternative models with the focus being on an n-variable product with zero or n responsibility centers. We supported the proposed models with a mathematical proof showing the difference of a product of n variables can be expressed as simply as a function of averages and differences. Specific revenue and spending models were presented.

REFERENCES

Aumann, R.J. and Maschler, M. (1985), "Game theoretic analysis of a bankruptcy problem from the Talmud." *Journal of Economic Theory*, Vol. 36, pp 195-213. DOI: 10.1016/00220531(85)90102-4.

Sorochuk, C.M., Wilck, J.H. and Leemis, L.M. (2023), "Budget Variance Analysis for Various Numbers and Combinations of Responsibility Centers", forthcoming, *Management Accounting Quarterly.*

Kloock, J. and Schiller, U. (1997), " Marginal costing: cost budgeting and cost variance analysis", *Management Accounting Research*, Vol. 8 No. 3, pp 299–323. DOI: 10.1006/mare.1996.0048.

Mitchell, T. and Thomas, M.F. (2005), "Can variance analysis make media marketing managers more accountable?", *Management Accounting Quarterly*, Vol. 7 No. 1, pp. 51-61.

Spiro, R., Stanton, W. and Rich, G. (2003), "*Management of a Sales Force, 11th Edition",* McGraw–Hill Irwin (Boston).

APPENDIX: PROOF OF THEOREM 1

For the purposes of this proof, let $x_{i,1} = b_i$ and $x_{i,2} = a_i$, for $i = 1, 2, \ldots, n$. It is sufficient to show that

$$\prod_{i=1}^{n} x_{i,2} - \prod_{i=1}^{n} x_{i,1} = \sum_{d=1, d\ odd}^{n} \frac{1}{2^{d-1}} \sum_{S \subseteq [n], |S|=d} \prod_{j \in S, k \in S'} (a_j - b_j)(a_k + b_k), \quad (4)$$

so that the right-hand side of the equation will be written entirely in terms of sums and differences. The result is proven by showing that of all of the monomials resulting by multiplying out the right-hand side of this equation, for example, $a_1 a_2 b_3 a_4 \ldots b_n$, all terms cancel except for $\prod_{i=1}^{n} a_i$ and $\prod_{i=1}^{n} b_i$. For the terms to cancel, there must be an equal number of positive and negative terms comprising the monomial. The number of odd-order subsets of $[n]$ is 2^{n-1}. This can be seen by expanding $(1-1)^n$ by the binomial theorem:

$$(1-1)^n \sum_{i=0}^{n} (-1)^n \binom{n}{i} = 0$$

Separating the negative and positive terms,

$$\sum_{i=0, i\ odd}^{n} \binom{n}{i} = \sum_{i=0, i\ even}^{n} \binom{n}{i}$$

Since

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

there are 2^{n-1} odd-ordered subsets and 2^{n-1} even-ordered subsets of $[n]$.

Now consider an arbitrary monomial resulting in multiplying out the terms on the right-hand side of Equation 4. To show that all terms except $\prod_{i=1}^{n} a_i$ and $\prod_{i=1}^{n} b_i$ cancel, consider the following cases involving a particularly arbitrary monomial other than $\prod_{i=1}^{n} a_i$ or $\prod_{i=1}^{n} b_i$.

Case 1. The number of times that the monomial is *negative* equals the number of odd-order subsets of $[n]$ containing an *odd* number of k-element index sets, which equals the product of the number of *odd*-order subsets of k-element sets and the number of *even*-order subsets of $(n-k)$-element complement sets, which, by the multiplication rule, is

$$2^{k-1} \cdot 2^{n-k-1} = 2^{n-2}.$$

Case 2. The number of times that the monomial is *positive* equals the number of odd- order subsets of $[n]$ containing an *even* number of k-element index sets, which equals the product of the number of *even*-order subsets of k-element sets and the number of *odd* -order subsets of $(n-k)$-element complement sets, which, by the multiplication rule, is

$$2^{k-1} \cdot 2^{n-k-1} = 2^{n-2}.$$

Since there are an equal number of positive and negative terms on any monomial term except $\prod_{i=1}^{n} a_i$ and $\prod_{i=1}^{n} b_i$, they must cancel. All of the 2^{n-1} products involving the monomial $a_1 a_2 \ldots a_n$ are positive and all of the 2^{n-1} products involving the monomial $b_1 b_2 \ldots b_n$ are negative, which proves the result.