Conjugacy Classes of the Symmetric Groups

Math 430 - Spring 2013

Understanding the conjugacy classes of a group G is an important part of understanding the group structure of G in general. Here, we determine the conjugacy classes of the symmetric group S_n. You may use these notes as a guide to Problem 8 in Section 37 (but write up a complete solution to it on your own).

We begin by noticing that any conjugate of a k-cycle is again a k-cycle.

Lemma 1. *Let $\alpha, \tau \in S_n$, where α is the k-cycle $(a_1\ a_2\ \cdots\ a_k)$. Then*

$$\tau\alpha\tau^{-1} = (\tau(a_1)\ \ \tau(a_2)\ \ \cdots\ \ \tau(a_k)).$$

Proof. Consider $\tau(a_i)$ such that $1 \leq i \leq k$. Then we have $\tau^{-1}\tau(a_i) = a_i$, and $\alpha(a_i) = a_{i+1 \bmod k}$. We now have $\tau\alpha\tau^{-1}(\tau(a_i)) = \tau(a_{i+1 \bmod k})$. Now take any j such that $j \in \{1, 2, 3, \ldots, n\}$, but $j \neq a_i$ for any i. Then $\alpha(j) = j$ since j is not in the k-cycle defining α. So, $\tau\alpha\tau^{-1}(\tau(j)) = \tau(j)$. We now see that $\tau\alpha\tau^{-1}$ fixes any number which is not of the form $\tau(a_i)$ for some i, and we have

$$\tau\alpha\tau^{-1} = (\tau(a_1)\ \ \tau(a_2)\ \ \cdots\ \ \tau(a_k)). \qquad \square$$

For any permutation $\alpha \in S_n$, we know we can write α as a product of disjoint cycles. Suppose we write α in this way, and α has cycles of length k_1, k_2, k_3, …, k_ℓ, where $k_1 \geq k_2 \geq k_3 \geq \ldots \geq k_\ell$, and where we include 1's in this list for fixed points. We call the sequence $(k_1, k_2, k_3, \ldots, k_\ell)$ the *cycle type* of α. Note that $\sum_{i=1}^{\ell} k_i = n$ since every element in $\{1, 2, \ldots, n\}$ is either fixed or appears in some cycle.

Example 1. If $\sigma \in S_{10}$ and $\sigma = (1\ 3\ 4\ 5)(2\ 7\ 8\ 9)$, then σ has cycle type $(4, 4, 1, 1)$.

Example 2. If α is a k-cycle in S_n, where $k \leq n$, then the cycle type of α is $(k, 1, \ldots, 1)$, where there are $n-k$ 1's in the sequence.

We may now describe the conjugacy classes of the symmetric groups.

Theorem 1. *The conjugacy classes of any S_n are determined by cycle type. That is, if σ has cycle type $(k_1, k_2, \ldots, k_\ell)$, then any conjugate of σ has cycle type $(k_1, k_2, \ldots, k_\ell)$, and if ρ is any other element of S_n with cycle type $(k_1, k_2, \ldots, k_\ell)$, then σ is conjugate to ρ.*

Proof. Suppose that σ has cycle type $(k_1, k_2, \ldots, k_\ell)$, so that σ can be written as a product of disjoint cycles as $\sigma = \alpha_1\alpha_2\cdots\alpha_\ell$, where α_i is a k_i-cycle. Let $\tau \in S_n$, then we have

$$\tau\sigma\tau^{-1} = \tau\alpha_1\alpha_2\cdots\alpha_\ell\tau^{-1} = (\tau\alpha_1\tau^{-1})(\tau\alpha_2\tau^{-1})\cdots(\tau\alpha_\ell\tau^{-1}). \tag{1}$$

Now, for each i such that $1 \leq i \leq \ell$, we have α_i is a k_i-cycle. From Lemma 1, we know that $\tau\alpha_i\tau^{-1}$ is also a k_i-cycle. For any $i, j \in \{1, 2, \ldots, \ell\}$ such that $i \neq j$, we know that α_i and α_j are disjoint, and so $\tau\alpha_i\tau^{-1}$ and $\tau\alpha_j\tau^{-1}$ must be disjoint since τ is a one-to-one function. So, the product in (1) above is $\tau\sigma\tau^{-1}$ written as a product of disjoint cycles, and $\tau\alpha_i\tau^{-1}$ is a k_i-cycle. Now we see that any conjugate of σ has cycle type $(k_1, k_2, \ldots, k_\ell)$.

Now let $\sigma, \rho \in S_n$ both be of cycle type $(k_1, k_2, \ldots, k_\ell)$, and we show that σ and ρ are conjugate in S_n. Let σ and τ be written as products of disjoint cycles as

$$\sigma = \alpha_1\alpha_2\cdots\alpha_\ell \quad \text{and} \quad \rho = \beta_1\beta_2\cdots\beta_\ell,$$

where α_i and β_i are k_i-cycles. For each i, let us write

$$\alpha_i = (a_{i1}\ a_{i2}\ \cdots\ a_{ik_i}) \quad \text{and} \quad \beta_i = (b_{i1}\ b_{i2}\ \cdots\ b_{ik_i}).$$

Now define τ by $\tau(a_{ij}) = b_{ij}$ for every i, j such that $1 \leq i \leq \ell$ and $1 \leq j \leq k_i$. From Lemma 1, we have $\tau\alpha_i\tau^{-1} = \beta_i$. So, from Exercise 1, we have

$$\tau\sigma\tau^{-1} = (\tau\alpha_1\tau^{-1})(\tau\alpha_2\tau^{-1})\cdots(\tau\alpha_\ell\tau^{-1}) = \beta_1\beta_2\cdots\beta_\ell = \rho.$$

So, any two elements of S_n with the same cycle type are in the same conjugacy class. □

If n is a positive integer, a sequence of positive integers $(k_1, k_2, \ldots, k_\ell)$ such that $k_1 \geq k_2 \geq \cdots \geq k_\ell$ and $\sum_{i=1}^{\ell} k_i = n$ is called a *partition* of n. From Theorem 1, the partitions of n are in one-to-one correspondence with the conjugacy classes of S_n. The number of partitions of a positive number n is often denoted $p(n)$, called the *partition function*, and we have $p(n)$ is the number of conjugacy classes of S_n.