Conjugacy Classes of the Symmetric Groups

Math 430 - Spring 2011

Let G be any group. If $g, x \in G$, we define the *conjugate of g by x* to be the element xgx^{-1}. (Note: some texts define the conjugate of g by x to be $x^{-1}gx$. By our definition, this would be the conjugate of g by x^{-1}.) If $g, h \in G$, and there is some $x \in G$ such that $xgx^{-1} = h$, we say that g and h are *conjugate* in G. For the group G, define the relation $\sim$ by $g \sim h$ if g and h are conjugate in G.

Proposition 1 *Let G be a group, and define the relation $\sim$ on G by $g \sim h$ if g and h are conjugate in G. Then $\sim$ is an equivalence relation on G.*

Proof. We need to check that $\sim$ satisfies the three defining properties of an equivalence relation. First, for any $g \in G$, we have $g \sim g$ since $ege^{-1} = g$, so the reflexive property holds. Now suppose $g\tilde{h}$. Then there is some $x \in G$ such that $xgx^{-1} = h$. Then we obtain $g = x^{-1}hx$. So we may conjugate h by x^{-1} to get g, so $h \sim g$ and the reflexive property holds. Now suppose $g, h, k \in G$, where $g \sim h$ and $h \sim k$. Then there are $y, z \in G$ such that $ygy^{-1} = h$ and $zhz^{-1} = k$. Substituting the former expression for h into the latter, we obtain $zygy^{-1}z^{-1} = k$, or $(zy)g(zy)^{-1} = k$. So, we may conjugate g by zy to get k, so $g \sim k$ and the transitive property holds. Thus $\sim$ is an equivalence relation on G. $\square$

Since $\sim$ is an equivalence relation on G, its equivalence classes partition G. The equivalence classes under this relation are called the *conjugacy classes* of G. So, the conjugacy class of $g \in G$ is

$$[g] = \{xgx^{-1} \mid x \in G\}.$$

Exercise 1. Let G be any group, and let $x, g_1, g_2, \ldots, g_n \in G$. Show that for any n, the conjugate of $g_1 g_2 \cdots g_n$ by x is the product of the conjugates by x of $g_1, g_2, \ldots, g_n$.

Exercise 2. Let G be an Abelian group. Show that for any $a \in G$, the conjugacy class of a is the singleton set $\{a\}$.

When G is non-Abelian, understanding the conjugacy classes of G is an important part of understanding the group structure of G. Conjugacy classes play a key role in a subject called *representation theory*, which is one of the main applications of group theory to chemistry and physics.

We now determine the conjugacy classes of the symmetric group S_n. We begin by noticing that any conjugate of a k-cycle is again a k-cycle.

Lemma 1 *Let $\alpha, \tau \in S_n$, where α is the k-cycle $(a_1\ a_2\ \cdots\ a_k)$. Then*

$$\tau\alpha\tau^{-1} = (\tau(a_1)\ \ \tau(a_2)\ \ \cdots\ \ \tau(a_k)).$$

Proof. Consider $\tau(a_i)$ such that $1 \leq i \leq k$. Then we have $\tau^{-1}\tau(a_i) = a_i$, and $\alpha(a_i) = a_{i+1 \bmod k}$. We now have $\tau\alpha\tau^{-1}(\tau(a_i)) = \tau(a_{i+1 \bmod k})$. Now take any j such that $j \in \{1, 2, 3, \ldots, n\}$, but $j \neq a_i$ for any i. Then $\alpha(j) = j$ since j is not in the k-cycle defining α. So, $\tau\alpha\tau^{-1}(\tau(j)) = \tau(j)$. We now see that $\tau\alpha\tau^{-1}$ fixes any number which is not of the form $\tau(a_i)$ for some i, and we have

$$\tau\alpha\tau^{-1} = (\tau(a_1)\ \ \tau(a_2)\ \ \cdots\ \ \tau(a_k)). \quad \square$$

For any permutation $\alpha \in S_n$, we know we can write α as a product of disjoint cycles. Suppose we write α in this way, and α has cycles of length k_1, k_2, k_3, $\ldots$, k_ℓ, where $k_1 \geq k_2 \geq k_3 \geq \ldots \geq k_\ell$, and where we include 1's in this list for fixed points. We call the sequence $(k_1, k_2, k_3, \ldots, k_\ell)$ the *cycle type* of α. Note that $\sum_{i=1}^{\ell} k_i = n$ since every element in $\{1, 2, \ldots, n\}$ is either fixed or appears in some cycle.

Example 1. If $\sigma \in S_{10}$ and $\sigma = (1\ 3\ 4\ 5)(2\ 7\ 8\ 9)$, then σ has cycle type $(4, 4, 1, 1)$.

Example 2. If α is a k-cycle in S_n, where $k \leq n$, then the cycle type of α is $(k, 1, \ldots, 1)$, where there are $n - k$ 1's in the sequence.

We may now describe the conjugacy classes of the symmetric groups.

Theorem 1 *The conjugacy classes of any S_n are determined by cycle type. That is, if σ has cycle type $(k_1, k_2, \ldots, k_\ell)$, then any conjugate of σ has cycle type $(k_1, k_2, \ldots, k_\ell)$, and if ρ is any other element of S_n with cycle type $(k_1, k_2, \ldots, k_\ell)$, then σ is conjugate to ρ.*

Proof. Suppose that σ has cycle type $(k_1, k_2, \ldots, k_\ell)$, so that σ can be written as a product of disjoint cycles as $\sigma = \alpha_1\alpha_2 \cdots \alpha_\ell$, where α_i is a k_i-cycle. Let $\tau \in S_n$, then by Exercise 1 we have

$$\tau\sigma\tau^{-1} = \tau\alpha_1\alpha_2 \cdots \alpha_\ell\tau^{-1} = (\tau\alpha_1\tau^{-1})(\tau\alpha_2\tau^{-1}) \cdots (\tau\alpha_\ell\tau^{-1}). \tag{1}$$

Now, for each i such that $1 \leq i \leq \ell$, we have α_i is a k_i-cycle. From Lemma 1, we know that $\tau\alpha_i\tau^{-1}$ is also a k_i-cycle. For any $i, j \in \{1, 2, \ldots, \ell\}$ such that $i \neq j$, we know that α_i and α_j are disjoint, and so $\tau\alpha_i\tau^{-1}$ and $\tau\alpha_j\tau^{-1}$ must be disjoint since τ is a one-to-one function. So, the product in (1) above is $\tau\sigma\tau^{-1}$ written as a product of disjoint cycles, and $\tau\alpha_i\tau^{-1}$ is a k_i-cycle. Now we see that any conjugate of σ has cycle type $(k_1, k_2, \ldots, k_\ell)$.

Now let $\sigma, \rho \in S_n$ both be of cycle type $(k_1, k_2, \ldots, k_\ell)$, and we show that σ and ρ are conjugate in S_n. Let σ and τ be written as products of disjoint cycles as

$$\sigma = \alpha_1\alpha_2 \cdots \alpha_\ell \quad \text{and} \quad \rho = \beta_1\beta_2 \cdots \beta_\ell,$$

where α_i and β_i are k_i-cycles. For each i, let us write

$$\alpha_i = (a_{i1}\ a_{i2}\ \cdots\ a_{ik_i}) \quad \text{and} \quad \beta_i = (b_{i1}\ b_{i2}\ \cdots\ b_{ik_i}).$$

Now define τ by $\tau(a_{ij}) = b_{ij}$ for every i, j such that $1 \leq i \leq \ell$ and $1 \leq j \leq k_i$. From Lemma 1, we have $\tau\alpha_i\tau^{-1} = \beta_i$. So, from Exercise 1, we have

$$\tau\sigma\tau^{-1} = (\tau\alpha_1\tau^{-1})(\tau\alpha_2\tau^{-1}) \cdots (\tau\alpha_\ell\tau^{-1}) = \beta_1\beta_2 \cdots \beta_\ell = \rho.$$

So, any two elements of S_n with the same cycle type are in the same conjugacy class. $\square$

If n is a positive integer, a sequence of positive integers $(k_1, k_2, \ldots, k_\ell)$ such that $k_1 \geq k_2 \geq \cdots \geq k_\ell$ and $\sum_{i=1}^{\ell} k_i = n$ is called a *partition* of n. From Theorem 1, the partitions of n are in one-to-one correspondence with the conjugacy classes of S_n. The number of partitions of a positive number n is often denoted $p(n)$, called the *partition function*, and we have $p(n)$ is the number of conjugacy classes of S_n. The partition function and its properties are of great interest in number theory. There is an extremely complicated formula for $p(n)$ in terms of n (involving an infinite sum, evaluations of derivatives of somewhat complicated functions, and curiously, the number 24 in various places), but there are several known simple modular arithmetic equivalences for the function. For example, if m is any non-negative integer, then it is known that $p(5m+4) \equiv 0 \bmod 5$.