The Commutator Subgroup

Math 430 - Spring 2011

Let G be any group. If $a, b \in G$, then the *commutator* of a and b is the element $aba^{-1}b^{-1}$. Of course, if a and b commute, then $aba^{-1}b^{-1} = e$. Now define C to be the set

$$C = \{x_1 x_2 \cdots x_n \mid n \geq 1, \text{ each } x_i \text{ is a commutator in } G\}.$$

In other words, C is the collection of all finite products of commutators in G. Then we have

Proposition 1. *If G is any group, then $C \lhd G$.*

Proof. First, we have $e = eee^{-1}e^{-1} \in C$, so C is nonempty and contains the identity. If $c, d \in C$, then we have $c = x_1 x_2 \cdots x_n$ and $d = y_1 y_2 \cdots y_m$, where each x_i and each y_j is a commutator in G. Then

$$cd = x_1 x_2 \cdots x_n y_1 y_2 \cdots y_m \in C,$$

since this is just another finite product of commutators. We also have

$$d^{-1} = (x_1 x_2 \cdots x_n)^{-1} = x_n^{-1} \cdots x_2^{-1} x_1^{-1}.$$

If $x_i = a_i b_i a_i^{-1} b_i^{-1}$, then $x_i^{-1} = b_i a_i b_i^{-1} a_i^{-1}$, which is also a commutator. Thus $c^{-1} \in C$, and $C \leq G$.

To prove C is a normal subgroup of G, let $g \in G$, and $c = x_1 x_2 \cdots x_n \in C$. Then we have

$$gcg^{-1} = gx_1 x_2 \cdots x_n g^{-1} = (gx_1 g^{-1})(gx_2 g^{-1}) \cdots (gx_n g^{-1}), \tag{1}$$

where we have just inserted $gg^{-1} = e$ between x_i and x_{i+1} for each $i < n$. Now, if $x_i = a_i b_i a_i^{-1} b_i^{-1}$, then we have

$$gx_i g^{-1} = ga_i b_i a_i^{-1} b_i^{-1} g^{-1} = (ga_i g^{-1})(gb_i g^{-1})(ga_i^{-1} g^{-1})(gb_i^{-1} g^{-1}).$$

Now note that $(ga_ig^{-1})^{-1} = (g^{-1})^{-1}a_i^{-1}g^{-1} = ga_i^{-1}g^{-1}$, and we have the analogous statement if we replace a_i by b_i. So, we have

$$gx_ig^{-1} = (ga_ig^{-1})(gb_ig^{-1})(ga_ig^{-1})^{-1}(gb_ig^{-1})^{-1},$$

which is a commutator. Now, from (1), we have gcg^{-1} is a product of commutators, and so $gcg^{-1} \in C$. Thus $C \lhd G$. $\square$

The subgroup C of G is called the *commutator subgroup* of G, and it general, it is also denoted by $C = G'$ or $C = [G, G]$, and is also called the *derived subgroup* of G. If G is Abelian, then we have $C = \{e\}$, so in one sense the commutator subgroup may be used as one measure of how far a group is from being Abelian. Specifically, we have the following result.

Theorem 1. *Let G be a group, and let C be its commutator subgroup. Suppose that $N \lhd G$. Then G/N is Abelian if and only if $C \subseteq N$. In particular, G/C is Abelian.*

Proof. First assume that G/N is Abelian. Let $a, b \in G$. Since we are assuming that G/N is Abelian, then we have $(aN)(bN) = (bN)(aN)$, and so $abN = baN$ by the definition of coset multiplication in the factor group. Now, we know $abN = baN$ implies $ab(ba)^{-1} \in N$, where $ab(ba)^{-1} = aba^{-1}b^{-1}$, and so $aba^{-1}b^{-1} \in N$. Since a and b were arbitrary, any commutator in G is an element of N, and since N is a subgroup of G, then any finite product of commutators in G is an element of N. Thus $C \subseteq N$.

Now suppose that $C \subseteq N$, and let $a, b \in G$. Then $aba^{-1}b^{-1} \in N$, and so $ab(ba)^{-1} \in N$. This implies $abN = baN$, or that $(aN)(bN) = (bN)(aN)$. Since a and b were arbitrary, this holds for any elements $aN, bN \in G/N$, and thus G/N is Abelian. $\square$