Quiz 5 **Solutions**, Math 211, Section 1 (Vinroot)

(a): Given that $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$, find the matrix C which satisfies $[\mathbf{v}]_{\mathcal{B}} = C\mathbf{v}$ for any $\mathbf{v}$ in $\mathbb{R}^2$.

Solution: We know that for any $\mathbf{v}$ in $\mathbb{R}^2$, we have $P_{\mathcal{B}}[\mathbf{v}]_{\mathcal{B}} = \mathbf{v}$ where $P_{\mathcal{B}} = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$. Then $[\mathbf{v}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\mathbf{v}$, and $P_{\mathcal{B}}$ is invertible since its columns are linearly independent (we are given that $\mathcal{B}$ is a basis). So, the C which we seek is $P_{\mathcal{B}}^{-1}$, and we have

$$C = P_{\mathcal{B}}^{-1} = \frac{1}{2 \cdot 3 - 1 \cdot 4} \begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3/2 & -2 \\ -1/2 & 1 \end{bmatrix}.$$

(b): Suppose that A is a matrix which has the following matrix as its reduced row echelon form:

$$\begin{bmatrix} 1 & 7 & 0 & 0 & 4 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

What are the dimensions of Col A and Nul A? Give a brief explanation based on the reduced row echelon form of A.

Solution: We know that the dimension of Col A is the number of pivot columns of A, which we see from the reduced row echelon from is 2. In particular, the first and third columns are pivot columns. Thus dim Col $A = 2$. We know that the dimension of Nul A is exactly the number of free variables in $A\mathbf{x} = \mathbf{0}$, and the free variables correspond to the columns of A which do not have pivots. Again from the reduced row echelon form given, we see that there will be 3 free variables, corresponding to the second, fourth, and fifth columns of A. Thus dim Nul $A = 3$.