Quiz 1 **Solutions**, Math 211, Section 1 (Vinroot)

Consider the following three vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 3 \\ -5 \\ 6 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}.$$

Determine, with an explanation, whether every vector in $\mathbb{R}^3$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$. Is $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathbb{R}^3$?

Solution: If $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is an arbitrary vector in $\mathbb{R}^3$, the question of whether $\mathbf{b}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ is the same as asking whether there is a solution to the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b}$ for every possible $\mathbf{b} \in \mathbb{R}^3$. The solutions to the vector equation is equivalent to the linear system, or matrix equation, with corresponding augmented matrix

$$\begin{bmatrix} 1 & 3 & -1 & b_1 \\ -2 & -5 & 1 & b_2 \\ 2 & 6 & -1 & b_3 \end{bmatrix}.$$

To row reduce, we replace R2 with R2 + 2(R1), and we replace R3 with R3 − 2(R1). The result is

$$\begin{bmatrix} 1 & 3 & -1 & b_1 \\ 0 & 1 & -1 & b_2 + 2b_1 \\ 0 & 0 & 1 & b_3 - 2b_1 \end{bmatrix}.$$

We can already see that there is a pivot in every row of this matrix, and no pivot in the last column, which means in particular that there is no row of zeroes in the row reduced coefficient matrix. This means that the system is consistent for all possible values of b_1, b_2, and b_3. This, in turn, means that the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b}$ has solutions for every possible $\mathbf{b}$ in $\mathbb{R}^3$, and so every vector in $\mathbb{R}^3$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$. The answer to the last question is also "Yes", since $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is the collection of all possible linear combinations of $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$, which we have just seen is all of $\mathbb{R}^3$.