Quiz 5 **Solutions**, Math 111, Section 4 (Vinroot)

(a): If $f(x) = \sin(x)^{\sin(x)}$ compute $f'(x)$, showing steps clearly.

Solution: Since this function is of the form $g(x)^{h(x)}$, we take the natural logarithm of both sides first, which gives $\ln(f(x)) = \ln(\sin(x)^{\sin(x)}) = \sin(x)\ln(\sin(x))$. Now we take the derivative of both sides, noting that $\frac{d}{dx}(\ln(f(x))) = \frac{f'(x)}{f(x)}$ from the chain rule. This gives

$$\begin{aligned}\frac{f'(x)}{f(x)} &= \frac{d}{dx}(\sin(x)\ln(\sin(x))) \\ &= \cos(x)\ln(\sin(x)) + \sin(x)\cdot\frac{1}{\sin(x)}\cdot\cos(x) = \cos(x)\ln(\sin(x)) + \cos(x).\end{aligned}$$

We obtain the final answer by multiplying both sides by $f(x)$, so

$$f'(x) = f(x)\left(\cos(x)\ln(\sin(x)) + \cos(x)\right) = \sin(x)^{\sin(x)}\left(\cos(x)\ln(\sin(x)) + \cos(x)\right)$$

(b): A colony of bacteria cells is growing exponentially, and starts as a population of 1.1 million cells. After 30 minutes, there are 2.3 millions cells. Find a formula for the number of bacteria cells there are (in millions of cells) after t minutes. Leave your answer in terms of logarithms and/or exponentials, and show all steps of your solution clearly.

Solution: We let $P(t)$ be the population of the colony in millions of cells at time t in minutes. We take $t = 0$ to be the initial time, so that $P(0) = 1.1$. Since we are assuming exponential growth, we know that $P(t) = P(0)e^{kt} = 1.1e^{kt}$ for some k. We calculate k by using the fact that $P(30) = 2.3$. We have

$$P(30) = 2.3 = 1.1e^{30k} \quad\Longrightarrow\quad \frac{2.3}{1.1} = e^{30k} \quad\Longrightarrow\quad \ln\left(\frac{2.3}{1.1}\right) = 30k \quad\Longrightarrow\quad k = \frac{1}{30}\ln\left(\frac{2.3}{1.1}\right).$$

We substitue this into our equation for $P(t)$ and obtain the final equation

$$P(t) = 1.1e^{\frac{t}{30}\ln\left(\frac{2.3}{1.1}\right)}.$$