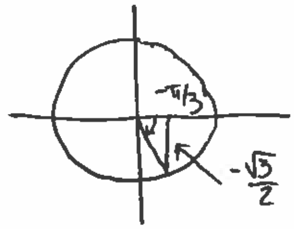
Math 103 - HW #8B

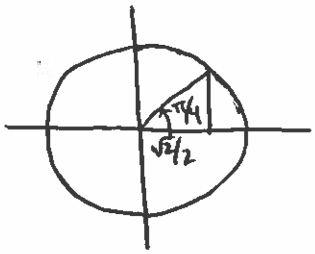
B1. (a)



Since $\sin\left(\frac{-\pi}{3}\right) = \frac{-\sqrt{3}}{2}$, then

$$\boxed{\arcsin\left(\frac{-\sqrt{3}}{2}\right) = \frac{-\pi}{3}}$$

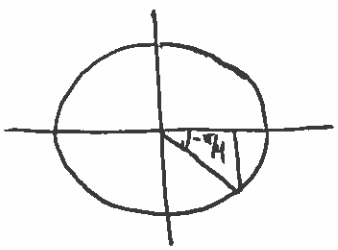
(b)

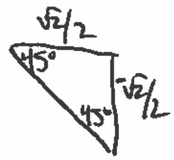


Since $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, then

$$\boxed{\arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}}$$

(c)



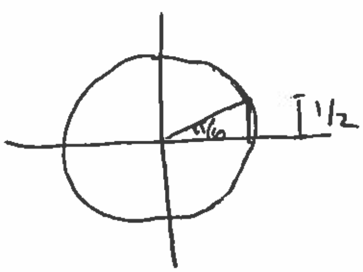


Since $\cos\left(\frac{-\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and $\sin\left(\frac{-\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

then $\tan\left(\frac{-\pi}{4}\right) = \frac{-\sqrt{2}/2}{\sqrt{2}/2} = -1.$

So $\boxed{\arctan(-1) = \frac{-\pi}{4}}$

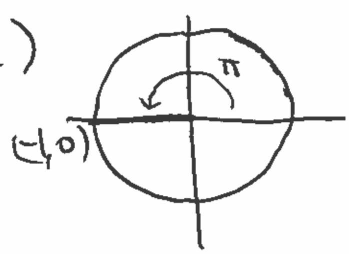
(d)



Since $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, then

$$\boxed{\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}}$$

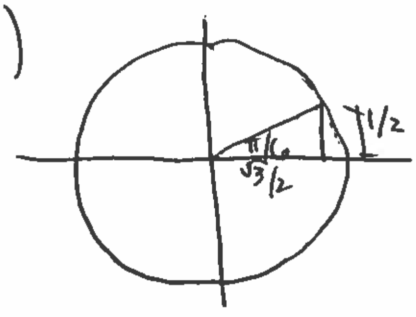
(e)



Since $\cos(\pi) = -1$, then

$$\boxed{\arccos(-1) = \pi}$$

(f)

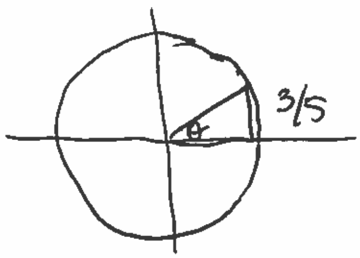


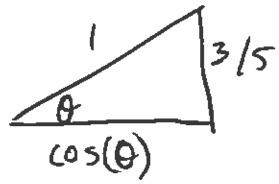
Since $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$,

then $\tan\left(\frac{\pi}{6}\right) = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

So $\boxed{\arctan\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}}$

B2. (a) If $\arcsin\left(\frac{3}{5}\right) = \theta$, then $\sin(\theta) = \frac{3}{5}$, and on the unit circle we have:





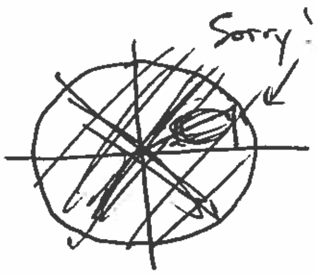
So $\cos^2(\theta) + \left(\frac{3}{5}\right)^2 = 1$,

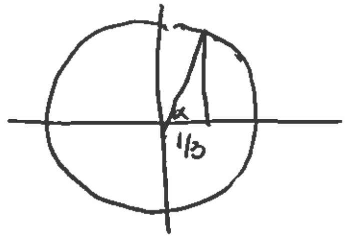
$\cos^2(\theta) = 1 - \frac{9}{25} = \frac{16}{25}$.

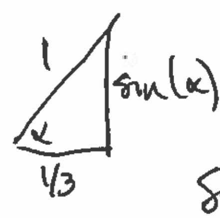
Since θ must be in the first quadrant from the range of arcsin, then $\cos(\theta) \geq 0$.

So $\cos(\theta) = \sqrt{\frac{16}{25}} = \frac{4}{5}$. Now $\cos(\theta) = \boxed{\cos\left(\arcsin\left(\frac{3}{5}\right)\right) = \frac{4}{5}}$

(b) If $\arccos\left(\frac{1}{3}\right) = \alpha$, then $\cos(\alpha) = \frac{1}{3}$, and α is in the first quadrant on the unit circle from the range of arccos. So we have:







$\sin^2(\alpha) + \left(\frac{1}{3}\right)^2 = 1$,

So $\sin^2(\alpha) = 1 - \frac{1}{9} = \frac{8}{9}$.

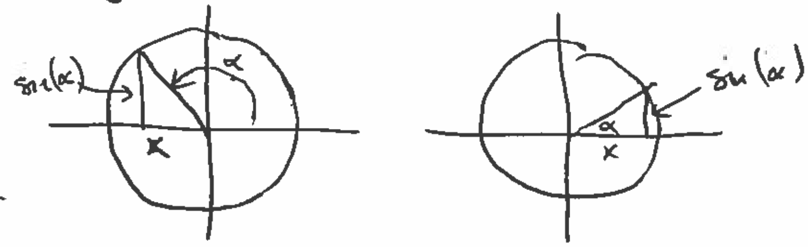
Since $\sin(\alpha) \geq 0$, we have $\sin(\alpha) = \sqrt{\frac{8}{9}} = \frac{\sqrt{8}}{3} = \frac{2\sqrt{2}}{3}$.

Now $\tan(\alpha) = \tan\left(\arccos\left(\frac{1}{3}\right)\right) = \frac{\sin(\alpha)}{\cos(\alpha)} = \frac{2\sqrt{2}/3}{1/3} = 2\sqrt{2}$.

So $\boxed{\tan\left(\arccos\left(\frac{1}{3}\right)\right) = 2\sqrt{2}}$

B3. If $\arccos(x) = \alpha$, then $\cos(\alpha) = x$, and we know $0 \leq \alpha \leq \pi$ from the range of arccos

For any angle in this range (between 0 and π), the value of $\sin(\alpha)$ is non-negative, which we can see from the unit circle:



In either case, we have

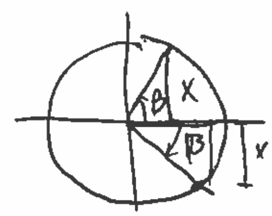
$\sin^2(\alpha) + \cos^2(\alpha) = 1$, so $\sin^2(\alpha) = 1 - \cos^2(\alpha) = 1 - x^2$.

Since $\sin(\alpha) \geq 0$, $\sin(\alpha) = \sqrt{1-x^2}$ (not the negative square root)

So $\sin(\alpha) = \boxed{\sin(\arccos(x)) = \sqrt{1-x^2}}$

If $\arcsin(x) = \beta$, then $-\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2}$ and $\sin(\beta) = x$.

Similar to the previous case, we know $\cos(\beta) \geq 0$ from looking at the unit circle:



$\cos(\beta) \geq 0$

Again, $\sin^2(\beta) + \cos^2(\beta) = 1$,

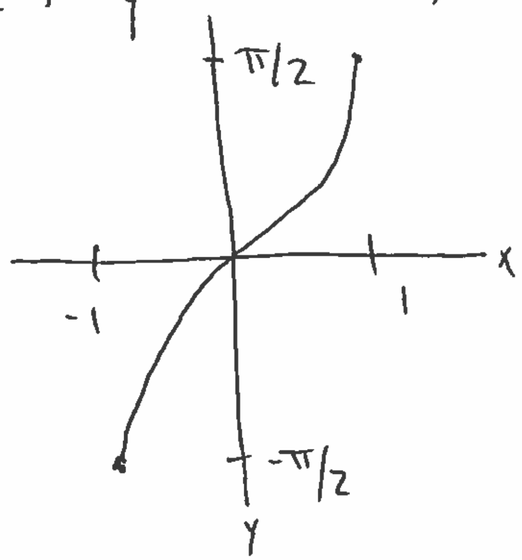
so $\cos^2(\beta) = 1 - \sin^2(\beta) = 1 - x^2$.

Since $\cos(\beta) \geq 0$, then $\cos(\beta) = \sqrt{1-x^2}$ (not the negative square root)

So $\cos(\beta) = \boxed{\cos(\arcsin(x)) = \sqrt{1-x^2}}$

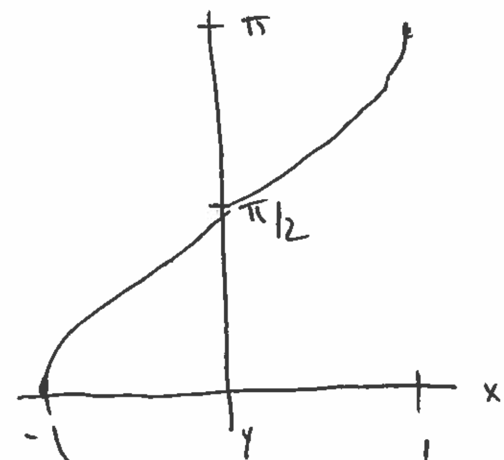
B4. In each of these, we can sketch the graph by shifting the graph of $y = \arcsin(x)$, $y = \arccos(x)$, or $y = \arctan(x)$ appropriately. We can then find the domain and range by reading this information from the graph.

(a) $y = \text{arsin}(x)$



$f(x) = \arcsin(x) + \frac{\pi}{2}$

shift up $\frac{\pi}{2}$ units:



$\boxed{\text{The domain is } -1 \le x \le 1}$,

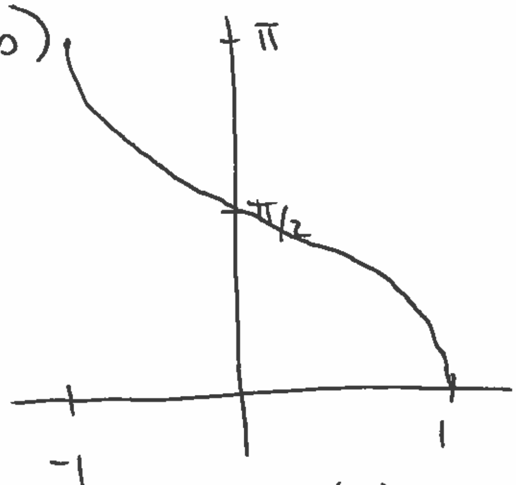
and the range is shifted up $\frac{\pi}{2}$ units,

so $\boxed{\text{the range is } 0 \le y \le \pi}$

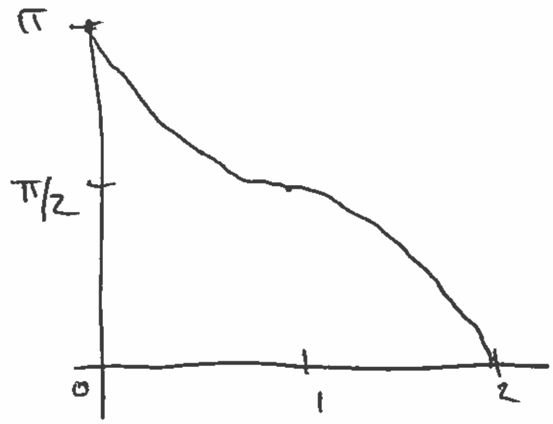
Note the graph of $y = \arcsin(x) + \frac{\pi}{2}$ is the same as the graph of $y = -\arccos(x)$. So $\arcsin(x) + \frac{\pi}{2}$
$= -\arccos(x)$

This is exactly because $\sin\left(\alpha + \frac{\pi}{2}\right) = -\cos(\alpha)$

B4 (b)



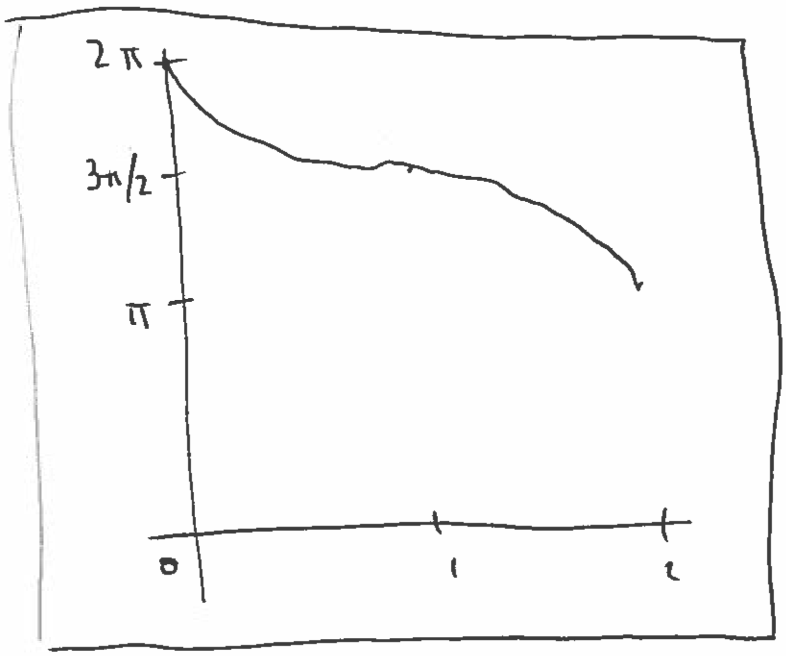
$y = \arccos(x)$



$y = \arccos(x-1)$: shift to the right by 1 unit

$g(x) = \arccos(x-1) + \pi$

Shift $y = \arccos(x-1)$ up π units:



From the graph,

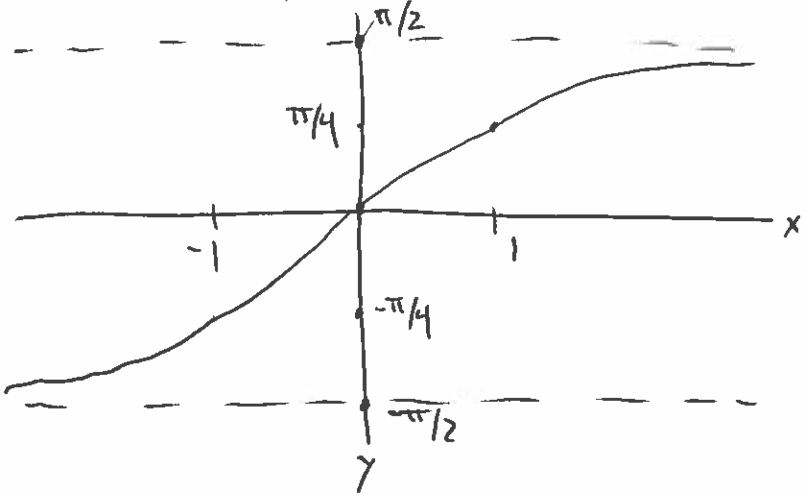
$\boxed{\text{the domain is}}$ $\boxed{0 \le x \le 2}$

(the domain of $y = \arccos(x)$ shifted one unit to the right),

and $\boxed{\text{the range is } \boxed{\pi \le y \le 2\pi}}$.

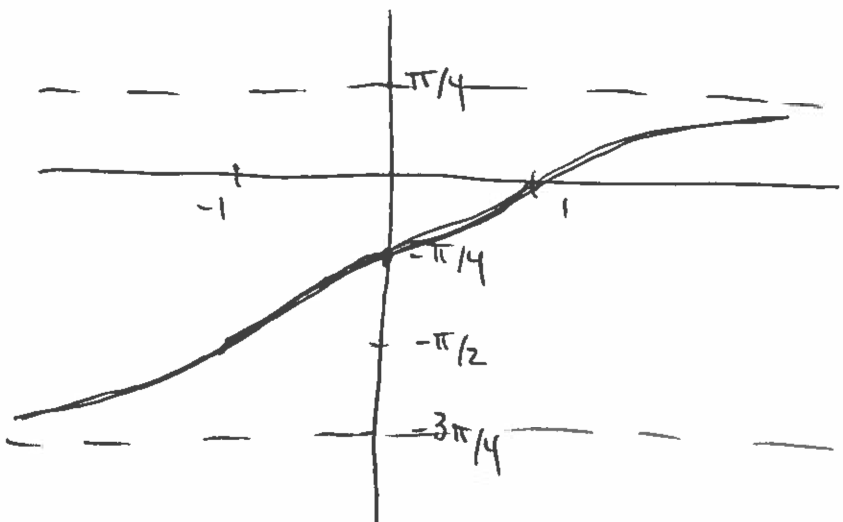
(the range of $y = \arccos(x)$ shifted up π units).

B4 (c) $y = \arctan(x)$

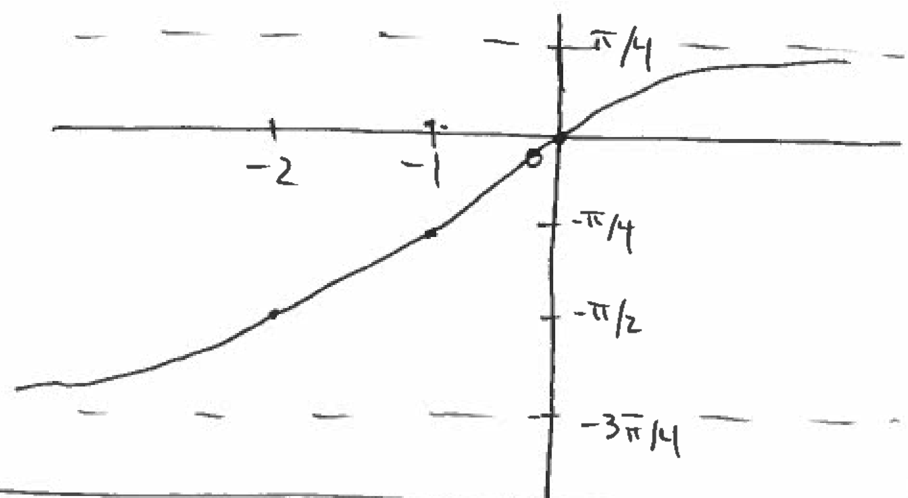


To graph $h(x) = \arctan(x+1) - \frac{\pi}{4}$, we shift this graph one unit to the left and $\frac{\pi}{4}$ units down.

$y = \arctan(x) - \frac{\pi}{4}$



$h(x) = \arctan(x+1) - \frac{\pi}{4}$

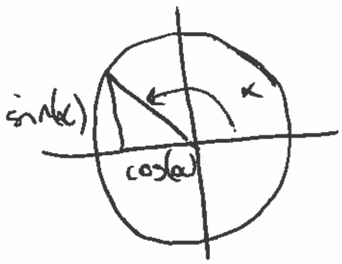


From the graph, $\boxed{\text{the domain is all real numbers}}$, and $\boxed{\text{the range is } -3\pi/4 < y < \pi/4}$

B5. If $\frac{\sin^2(\alpha)}{\cos^2(\alpha)} = \frac{16}{25}$, then $\left(\frac{\sin(\alpha)}{\cos(\alpha)}\right)^2 = \tan^2(\alpha) = \frac{16}{25}$.

Since $\frac{\pi}{2} < \alpha < \pi$, then from the unit circle, the angle is in the second quadrant, so

$\sin(\alpha) > 0$ and $\cos(\alpha) < 0$.



This means $\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)} < 0$.

So $\tan(\alpha) = -\sqrt{\frac{16}{25}} = \frac{-4}{5}$ since $\tan(\alpha)$ is negative.

Now $\cot(\alpha) = \frac{\cos(\alpha)}{\sin(\alpha)} = \frac{1}{\tan(\alpha)} = \frac{1}{-4/5} = \frac{-5}{4}$. So $\boxed{\cot(\alpha) = \frac{-5}{4}}$

B6. Since $\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$, then by using the angle sum formulas for sine and cosine, we have

$$\tan(\alpha+\beta) = \frac{\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)}.$$

Now divide the numerator and denominator by $\cos(\alpha)\cos(\beta)$:

$$\tan(\alpha+\beta) = \frac{\frac{\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}}{\frac{\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}} = \frac{\frac{\sin(\alpha)\cancel{\cos(\beta)}}{\cos(\alpha)\cancel{\cos(\beta)}} + \frac{\cancel{\cos(\alpha)}\sin(\beta)}{\cancel{\cos(\alpha)}\cos(\beta)}}{\frac{\cancel{\cos(\alpha)}\cancel{\cos(\beta)}}{\cancel{\cos(\alpha)}\cancel{\cos(\beta)}} - \frac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}}$$

$$= \frac{\frac{\sin(\alpha)}{\cos(\alpha)} + \frac{\sin(\beta)}{\cos(\beta)}}{1 - \frac{\sin(\alpha)}{\cos(\alpha)}\frac{\sin(\beta)}{\cos(\beta)}} = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}.$$

So $\boxed{\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}}$

B7. Letting $w = \cos(x)$, the equation is $2w^3 - 4w^2 - w + 2 = 0$.

Trying divisors of 2 as roots through synthetic division,

if we try 2 :

$$\begin{array}{r|rrrr} 2 & 2 & -4 & -1 & 2 \\ & & 4 & 0 & -2 \\ \hline & 2 & 0 & -1 & \boxed{0} \end{array}$$

We find $w-2$ is a factor, and the polynomial factors

as $\quad 2w^3 - 4w^2 - w + 2 = (w-2)(2w^2-1) = 0$.

So $\quad w - 2 = 0 \quad$ or $\quad 2w^2 - 1 = 0$, so

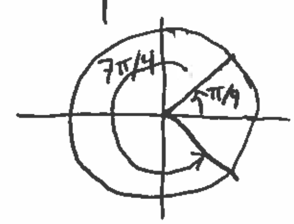
$\quad w = 2 \quad$ or $\quad 2w^2 = 1$, so $w^2 = \frac{1}{2}$, so $w = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$.

Now substitute back $w = \cos(x)$. Then $\cos(x) = 2$

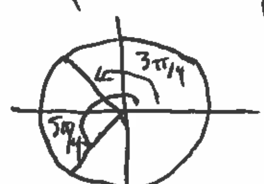
has no solutions, since $-1 \le \cos(x) \le 1$ for all real x.

Now $w = \frac{\pm\sqrt{2}}{2}$ means $\cos(x) = \frac{\sqrt{2}}{2}$ or $\cos(x) = -\frac{\sqrt{2}}{2}$.

If $0 \le x \le 2\pi$, then $\cos(x) = \frac{\sqrt{2}}{2}$ when $x = \frac{\pi}{4}$ or $x = \frac{7\pi}{4}$,



and $\cos(x) = -\frac{\sqrt{2}}{2}$ when $x = \frac{3\pi}{4}$ or $x = \frac{5\pi}{4}$



So all values of x

such that $\cos(x) = \frac{\pm\sqrt{2}}{2}$

are given by
$$\boxed{x = \frac{\pi}{4} + 2\pi k,\ \frac{7\pi}{4} + 2\pi k,\ \frac{3\pi}{4} + 2\pi k,\ \frac{5\pi}{4} + 2\pi k \text{ for } k \text{ any integer,}}$$

and these are all of the solutions to the original equation.