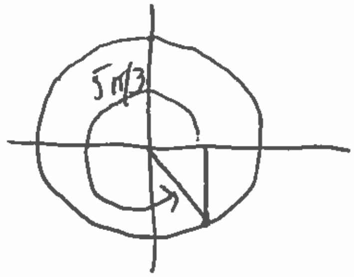
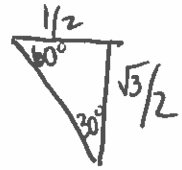
Math 103 - HW #7A, B

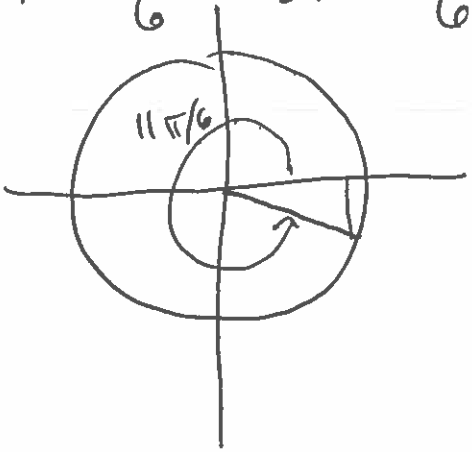
A1. (a) $\frac{5\pi}{3} = 2\pi - \frac{\pi}{3}$

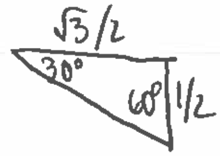




$\sin\left(\frac{5\pi}{3}\right) = \boxed{\frac{-\sqrt{3}}{2}}$

(b) $\frac{11\pi}{6} = 2\pi - \frac{\pi}{6}$

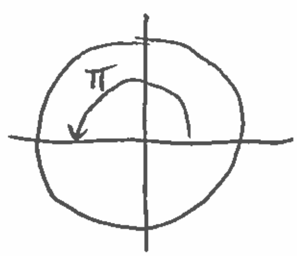




$\csc\left(\frac{11\pi}{6}\right) = \frac{1}{\sin\left(\frac{11\pi}{6}\right)}$

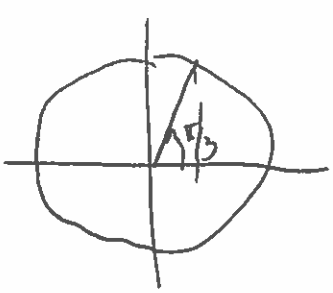
$= \frac{1}{-1/2} = \boxed{-2}$

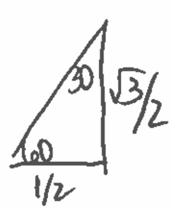
(c) $7\pi = (2\pi)3 + \pi$, so $\cot(7\pi) = \cot(\pi) = \frac{\cos(\pi)}{\sin(\pi)}$



But $\sin(\pi) = 0$, so
$\cot(\pi)$ is $\boxed{\text{undefined}}$

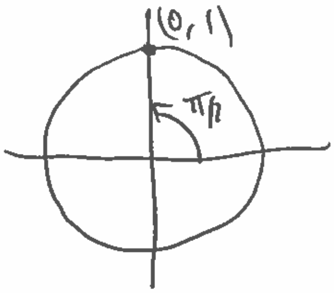
(d) $\tan(\pi/3) = \frac{\sin(\pi/3)}{\cos(\pi/3)} = \frac{\sqrt{3}/2}{1/2} = \boxed{\sqrt{3}}$



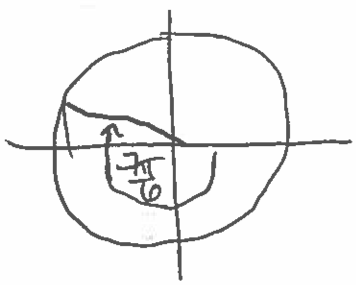


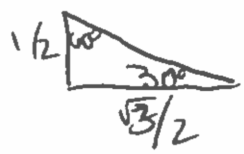
A1 (e) $\frac{21\pi}{2} = \frac{20\pi}{2} + \frac{\pi}{2} = 10\pi + \frac{\pi}{2} = (2\pi)5 + \frac{\pi}{2}$,

so $\cos\left(\frac{21\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right)$, so $\cos\left(\frac{21\pi}{2}\right) = \boxed{0}$



(f) $-\frac{7\pi}{6} = -\frac{6\pi}{6} - \frac{\pi}{6} = -\left(\pi + \frac{\pi}{6}\right)$





$$\sec\left(\frac{-7\pi}{6}\right) = \frac{1}{\cos\left(\frac{-7\pi}{6}\right)} = \frac{1}{-\frac{\sqrt{3}}{2}}$$

$$= \boxed{\frac{-2}{\sqrt{3}} = \frac{-2\sqrt{3}}{3}}$$

A2. $\tan(x) + \cot(x) = \frac{\sin(x)}{\cos(x)} + \frac{\cos(x)}{\sin(x)} = \frac{\sin(x)\cdot\sin(x)}{\cos(x)\cdot\sin(x)} + \frac{\cos(x)\cdot\cos(x)}{\sin(x)\cdot\cos(x)}$

$$= \frac{\sin^2(x) + \cos^2(x)}{\sin(x)\cos(x)} = \frac{1}{\sin(x)\cos(x)} = \frac{1}{\cos(x)} \cdot \frac{1}{\sin(x)}$$

(since $\sin^2(x) + \cos^2(x) = 1$)

(get a common denominator)

$= \sec(x)\csc(x)$.

So $\tan(x) + \cot(x) = \sec(x)\csc(x)$.

since $\sec(x) = \frac{1}{\cos(x)}$ and $\csc(x) = \frac{1}{\sin(x)}$.

A3. $\sin^2(\theta)\csc^2(\theta) - \sin^2(\theta) = \sin^2(\theta)\cdot\frac{1}{\sin^2(\theta)} - \sin^2(\theta)$

since $\csc(\theta) = \frac{1}{\sin(\theta)}$

$= 1 - \sin^2(\theta) = \cos^2(\theta)$

since $\sin^2(\theta) + \cos^2(\theta) = 1$

So $\sin^2(\theta)\csc^2(\theta) - \sin^2(\theta) = \cos^2(\theta)$.

A4. Since $\sin^2(x) + \cos^2(x) = 1$, $\sin^2(x) = 1 - \cos^2(x)$.

So $\sin^4(x) = \sin^2(x)\cdot\sin^2(x) = (1-\cos^2(x))(1-\cos^2(x)) = (1-\cos^2(x))^2$. Now we have

$\sin^4(x) + \cos^4(x) = (1-\cos^2(x))^2 + \cos^4(x)$

Multiply this out

$= (1 - 2\cos^2(x) + \cos^4(x)) + \cos^4(x)$

$= 1 - 2\cos^2(x) + 2\cos^4(x)$.

So $\sin^4(x) + \cos^4(x) = 1 - 2\cos^2(x) + 2\cos^4(x)$.

A5. (a) Start with $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$.

Letting $\alpha = \beta$ gives

$\sin(\alpha+\alpha) = \sin(\alpha)\cos(\alpha) + \cos(\alpha)\sin(\alpha)$, so

$\boxed{\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha).}$

A5 (b): Starting with $\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$

let $\beta=\alpha$, so $\cos(\alpha+\alpha)=\cos(\alpha)\cos(\alpha)-\sin(\alpha)\sin(\alpha)$,

so $\boxed{\cos(2\alpha)=\cos^2(\alpha)-\sin^2(\alpha)}$

Since $\cos^2(\alpha)=1-\sin^2(\alpha)$, then

$\cos^2(\alpha)-\sin^2(\alpha)=1-\sin^2(\alpha)-\sin^2(\alpha)=1-2\sin^2(\alpha)$,

so also $\boxed{\cos(2\alpha)=1-2\sin^2(\alpha)}$.

Also $\sin^2(\alpha)=1-\cos^2(\alpha)$, so

$\cos^2(\alpha)-\sin^2(\alpha)=\cos^2(\alpha)-(1-\cos^2(\alpha))=\cos^2(\alpha)-1+\cos^2(\alpha)$

$=2\cos^2(\alpha)-1$. So we have

$\boxed{\cos(2\alpha)=2\cos^2(\alpha)-1}$

A6 (a) From A5(b), $\cos(2\alpha)=2\cos^2(\alpha)-1$. Solve for

$\cos(\alpha)$: $\qquad \cos(2\alpha)+1=2\cos^2(\alpha)$, so

$$\cos^2(\alpha)=\frac{\cos(2\alpha)+1}{2},$$ so

$$\cos(\alpha)=\sqrt{\frac{\cos(2\alpha)+1}{2}}=\sqrt{\frac{1+\cos(2\alpha)}{2}}$$

(Note there could also be a negative square root, but we'll only apply this when $\cos(\alpha)$ is positive).

Letting $\alpha=\frac{\theta}{2}$, so $2\alpha=\theta$, we have $\boxed{\cos\left(\frac{\theta}{2}\right)=\sqrt{\frac{1+\cos(\theta)}{2}}}$

A6 (b): From A5(b), we have $\cos(2\alpha) = 1 - 2\sin^2(\alpha)$.

Solve for $\sin(\alpha)$: $\cos(2\alpha) - 1 = -2\sin^2(\alpha)$

$$\frac{\cos(2\alpha)-1}{-2} = \sin^2(\alpha), \text{ so } \frac{1-\cos(2\alpha)}{2} = \sin^2(\alpha)$$

$(-(\cos(2\alpha)-1) = 1-\cos(2\alpha))$. Then $\sin(\alpha) = \sqrt{\dfrac{1-\cos(2\alpha)}{2}}$

(Again, we will only apply this when $\sin(\alpha)$ is positive, so we don't worry about the negative square root).

If $\alpha = \frac{\theta}{2}$, so $\theta = 2\alpha$, we have $\boxed{\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos(\theta)}{2}}}$

A7. (a) $\pi/8 = \dfrac{\pi/4}{2}$, so from A6(b), we have

$$\sin\left(\frac{\pi}{8}\right) = \sin\left(\frac{\pi/4}{2}\right) = \sqrt{\frac{1-\cos(\pi/4)}{2}} = \sqrt{\frac{1-\sqrt{2}/2}{2}}$$

$$= \sqrt{\frac{1}{2} - \frac{\sqrt{2}}{4}} = \sqrt{\frac{2-\sqrt{2}}{4}} = \boxed{\frac{\sqrt{2-\sqrt{2}}}{2}}$$

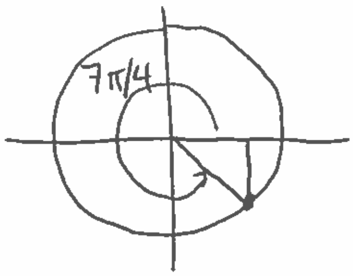
(b) $\dfrac{\pi}{12} = \dfrac{\pi/6}{2}$, so from A6(a), we have

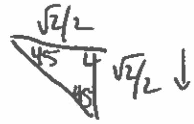
$$\cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi/6}{2}\right) = \sqrt{\frac{1+\cos(\frac{\pi}{6})}{2}} = \sqrt{\frac{1+\sqrt{3}/2}{2}} = \sqrt{\frac{1}{2} + \frac{\sqrt{3}}{4}}$$

$$= \sqrt{\frac{2+\sqrt{3}}{4}} = \boxed{\frac{\sqrt{2+\sqrt{3}}}{2}}$$

(This can be further simplified to $\dfrac{\sqrt{6}+\sqrt{2}}{4}$, but <u>not</u> in an obvious way).

B1. (a) $\frac{7\pi}{4} = 2\pi - \frac{\pi}{4}$

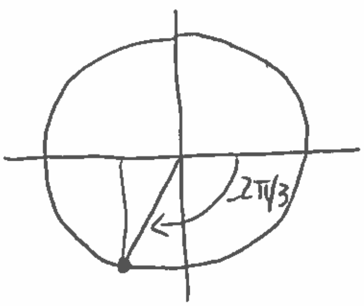


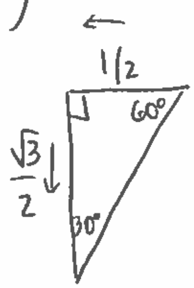


$$\csc\left(\frac{7\pi}{4}\right) = \frac{1}{\sin\left(\frac{7\pi}{4}\right)} = \frac{1}{-\frac{\sqrt{2}}{2}}$$

$$= \frac{-2}{\sqrt{2}} = \boxed{-\sqrt{2}}$$

(b) $\frac{-2\pi}{3} = -\left(\pi - \frac{\pi}{3}\right)$

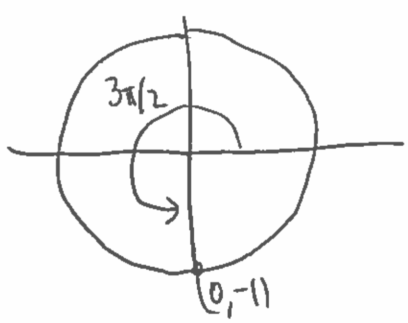




$$\cos\left(\frac{-2\pi}{3}\right) = \boxed{-\frac{1}{2}}$$

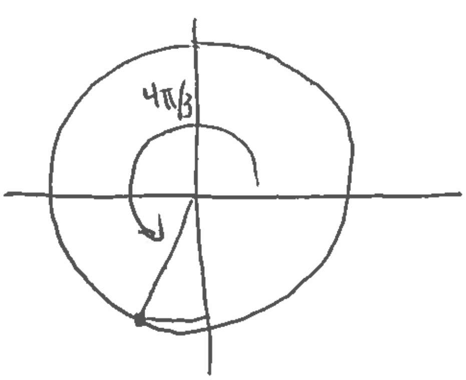
(c) $\frac{3\pi}{2} = \pi + \frac{\pi}{2}$

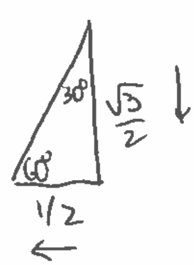
$$\cot\left(\frac{3\pi}{2}\right) = \frac{\cos(3\pi/2)}{\sin(3\pi/2)} = \frac{0}{-1} = \boxed{0}$$



(d) $\frac{4\pi}{3} = \pi + \frac{\pi}{3}$

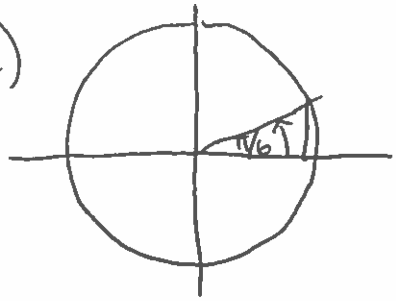
$$\tan\left(\frac{4\pi}{3}\right) = \frac{\sin(4\pi/3)}{\cos(4\pi/3)}$$

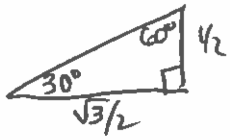




$$= \frac{-\sqrt{3}/2}{-1/2} = \boxed{\sqrt{3}}$$

B1 (e)

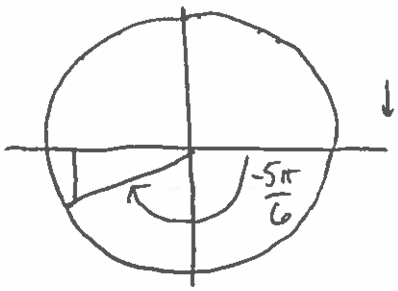


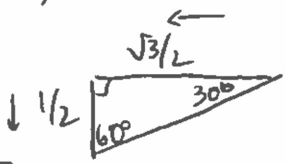


$$\sec\left(\frac{\pi}{6}\right) = \frac{1}{\cos\left(\frac{\pi}{6}\right)} = \frac{1}{\frac{\sqrt{3}}{2}}$$

$$= \boxed{\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}}$$

(f) $-\frac{5\pi}{6} = -\left(\pi - \frac{\pi}{6}\right)$





$$\sin\left(-\frac{5\pi}{6}\right) = \boxed{-\frac{1}{2}}$$

B2 (a) $\frac{1+\sin(x)}{\cos(x)} = \frac{1}{\cos(x)} + \frac{\sin(x)}{\cos(x)} = \sec(x) + \tan(x)$

(b) $\frac{\cot(x) - \tan(x)}{\cot(x) + \tan(x)} = \frac{\frac{\cos(x)}{\sin(x)} - \frac{\sin(x)}{\cos(x)}}{\frac{\cos(x)}{\sin(x)} + \frac{\sin(x)}{\cos(x)}}$ (Get common denominators)

$$= \frac{\frac{\cos(x)\cos(x)}{\sin(x)\cos(x)} - \frac{\sin(x)\sin(x)}{\cos(x)\sin(x)}}{\frac{\cos(x)\cos(x)}{\sin(x)\cos(x)} + \frac{\sin(x)\sin(x)}{\cos(x)\sin(x)}} = \frac{\frac{\cos^2(x) - \sin^2(x)}{\cancel{\sin(x)\cos(x)}}}{\frac{\cos^2(x) + \sin^2(x)}{\cancel{\sin(x)\cos(x)}}}$$

$= 1$

$= \cos^2(x) - \sin^2(x) = \cos(2x)$, From A5 (b)

(In the last step, you apply the double angle formula for cosine)

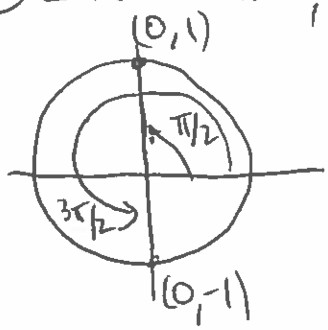
B3 (a) First, $3\cos^2(x) - \cos(2x) = 3\cos^2(x) - (2\cos^2(x) - 1)$

$= \cos^2(x) + 1$, so the equation becomes

$\cos^2(x) + 1 = 1$, so $\cos^2(x) = 0$. But if

$\cos^2(x) = (\cos(x))^2 = 0$, then $\cos(x) = 0$. If

$0 \le x < 2\pi$, then $\cos(x) = 0$ when $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.



So all angles x satisfying $\cos(x) = 0$, and so are solutions to our equation, are $x = \boxed{\frac{\pi}{2} + 2\pi k \text{ or } \frac{3\pi}{2} + 2\pi k, \text{ for any integer } k}$

(b) $5\cos(3x) + 5\sin(x)\cos(3x) = 0$, so

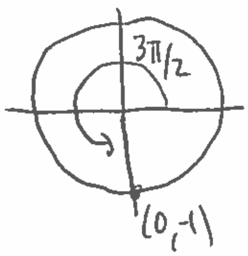
$5\cos(3x)(1 + \sin(x)) = 0$, so

$5\cos(3x) = 0$ or $1 + \sin(x) = 0$.

If $5\cos(3x) = 0$, then $\cos(3x) = 0$. If $0 \le 3x < 2\pi$, then $\cos(3x) = 0$ if $3x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

So the set of all angles $3x$ satisfying $\cos(3x) = 0$ are $3x = \frac{\pi}{2} + 2\pi k$ or $\frac{3\pi}{2} + 2\pi k$ for any integer k.

So when $x = \frac{\pi}{6} + \frac{2\pi}{3}k$ or $\frac{\pi}{2} + \frac{2\pi}{3}k$ for any integer k (by dividing by 3 on both sides). If $1 + \sin(x) = 0$, B3 (b) (cont'd) then $\sin(x) = -1$. If $0 \le x < 2\pi$,



$\sin(x) = -1$ only for $x = \frac{3\pi}{2}$. So all x satisfying $\sin(x) = -1$ are given by $x = \frac{3\pi}{2} + 2\pi k$ for any integer k.

So all solutions are given by:

$$\boxed{x = \frac{\pi}{6} + \frac{2\pi}{3}k, \ \frac{\pi}{2} + \frac{2\pi}{3}k, \text{ or } \frac{3\pi}{2} + 2\pi k, \text{ for } k \text{ any integer}}$$

B4 If $\cos(\theta) = \frac{2}{3}$, then since $\cos^2(\theta) + \sin^2(\theta) = 1$,

$$\sin^2(\theta) = 1 - \cos^2(\theta) = 1 - \left(\frac{2}{3}\right)^2 = 1 - \frac{4}{9} = \frac{5}{9}.$$

So $\sin(\theta) = \pm\sqrt{\frac{5}{9}} = \frac{\pm\sqrt{5}}{3}$. So $\boxed{\sin(\theta) = \frac{\sqrt{5}}{3} \text{ or } -\frac{\sqrt{5}}{3}}$

Then $\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{3}{2}$. $\boxed{\sec(\theta) = \frac{3}{2}}$

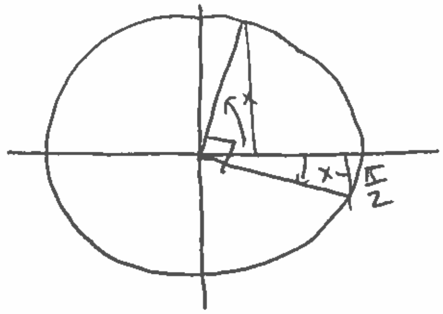
Then $\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{1}{\pm\sqrt{5}/3} = \frac{\pm 3}{\sqrt{5}}$. So $\boxed{\csc(\theta) = \frac{3}{\sqrt{5}} \text{ or } \frac{-3}{\sqrt{5}} = \frac{3\sqrt{5}}{5} \text{ or } \frac{-3\sqrt{5}}{5}}$

Then $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{\pm\sqrt{5}/3}{2/3} = \frac{\pm\sqrt{5}}{2}$.

So $\boxed{\tan(\theta) = \frac{\sqrt{5}}{2} \text{ or } -\frac{\sqrt{5}}{2}}$. Finally, $\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)} = \frac{1}{\tan(\theta)} =$

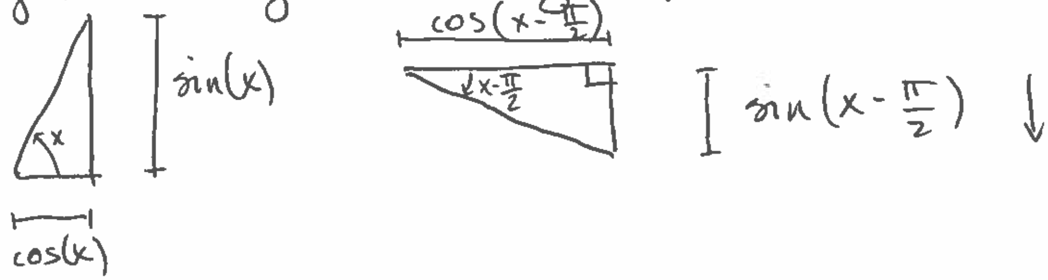
$= \frac{1}{\pm\sqrt{5}/2} = \frac{\pm 2}{\sqrt{5}} = \frac{\pm 2\sqrt{5}}{5}$. So $\boxed{\cot(\theta) = \frac{2}{\sqrt{5}} \text{ or } -\frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5} \text{ or } \frac{-2\sqrt{5}}{5}}$

B5 (a)



An arbitrary angle x on the unit circle is drawn, along with the angle $x-\frac{\pi}{2}$ (rotate clockwise 90°)

We get two right triangles with cosines and sines pictured.



Because we rotated by $\frac{\pi}{2}$ radians (90°) to get from angle x to angle $x-\frac{\pi}{2}$, these triangles are the <u>same triangle</u>, just oriented differently.

The first triangle has base $\cos(x)$, which is the same length as the vertical side of the second triangle. But this gives $\sin(x-\frac{\pi}{2})$, and goes in the negative vertical direction. This means

$$\boxed{\sin\left(x-\frac{\pi}{2}\right) = -\cos(x)}$$

. The height of the first triangle is $\sin(x)$, which is the same as the base of the second triangle, given by $\cos(x-\frac{\pi}{2})$.

These are both positive values in the picture, so

$$\boxed{\cos\left(x-\frac{\pi}{2}\right) = \sin(x)}$$

.

B5 (b): Now, $\tan\left(x-\frac{\pi}{2}\right)=\dfrac{\sin\left(x-\frac{\pi}{2}\right)}{\cos\left(x-\frac{\pi}{2}\right)}$. From (a),

this is $\dfrac{-\cos(x)}{\sin(x)}=-\cot(x)$. So $\tan\left(x-\frac{\pi}{2}\right)=-\cot(x)$,

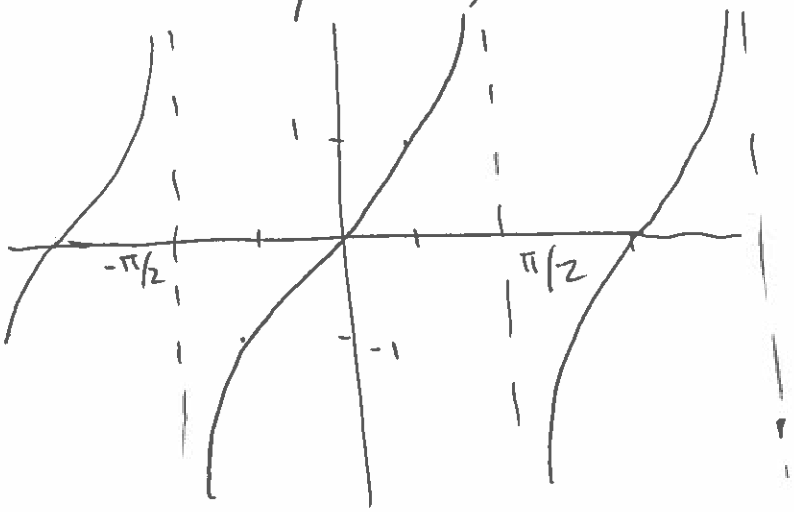
so $\boxed{-\tan\left(x-\frac{\pi}{2}\right)=\cot(x)}$. Then $\sec\left(x-\frac{\pi}{2}\right)=\dfrac{1}{\cos\left(x-\frac{\pi}{2}\right)}$

$=\dfrac{1}{\sin(x)}=\csc(x)$, so $\boxed{\sec\left(x-\frac{\pi}{2}\right)=\csc(x)}$.

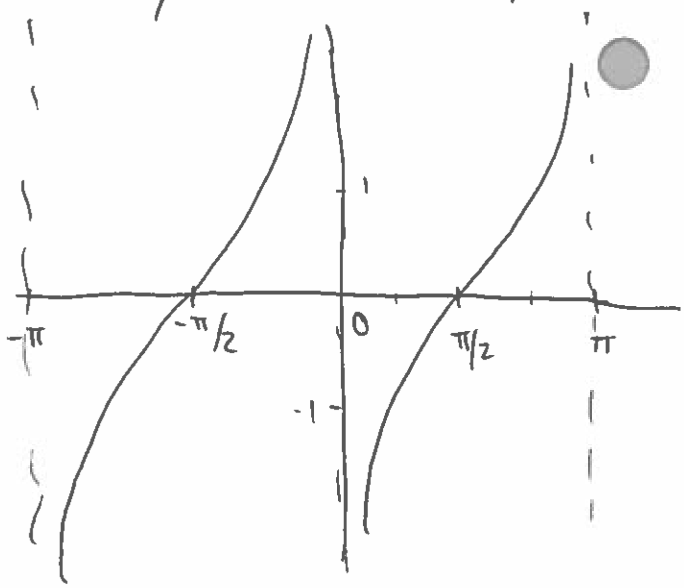
(c) Since $\cot(x)=-\tan\left(x-\frac{\pi}{2}\right)$, the graph of $y=\cot(x)=-\tan\left(x-\frac{\pi}{2}\right)$ is obtained by the graph of $y=\tan(x)$ by first shifting to the right by $\frac{\pi}{2}$ units to get the graph of $y=\tan\left(x-\frac{\pi}{2}\right)$, then vertically flipping to get the graph of $y=-\tan\left(x-\frac{\pi}{2}\right)$ (The graph of $y=-f(x)$ is obtained from the graph of $y=f(x)$ by vertically flipping the graph, or reflecting through the x-axis). Since $\csc(x)=\sec\left(x-\frac{\pi}{2}\right)$, the graph of $y=\csc(x)$ is obtained by shifting to the right $\frac{\pi}{2}$ units the graph of $y=\sec(x)$. These graphs are as follows:

B5 (c) (cont'd).

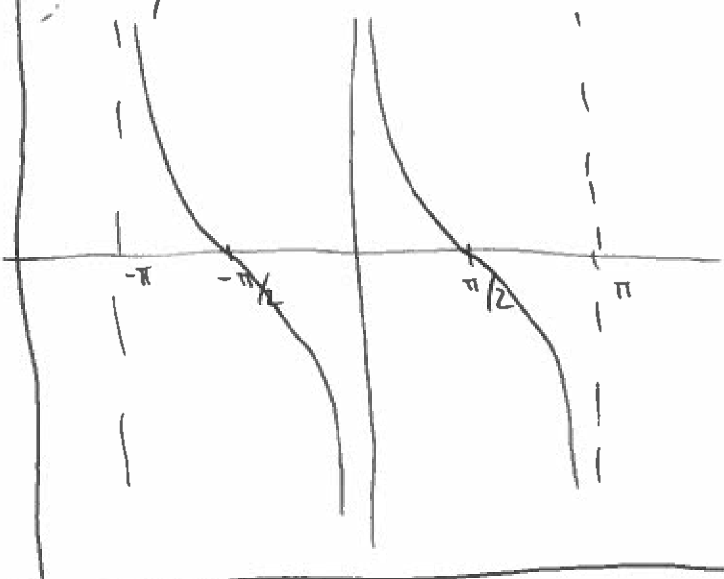
$y = \tan(x)$



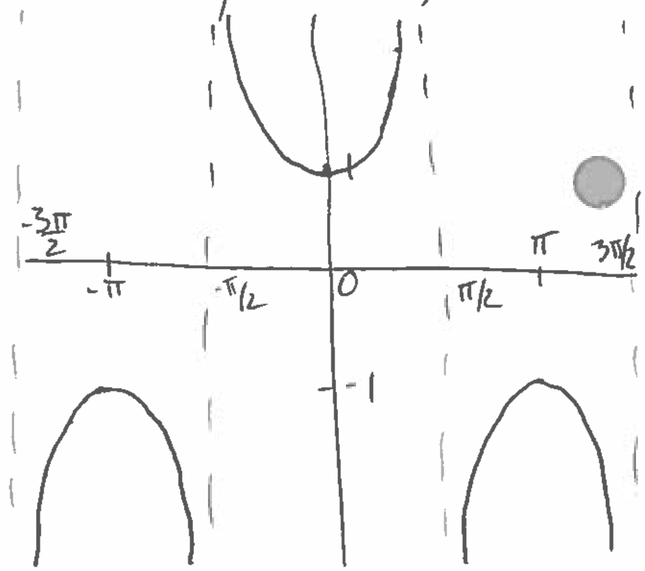
$y = \tan\left(x - \frac{\pi}{2}\right)$



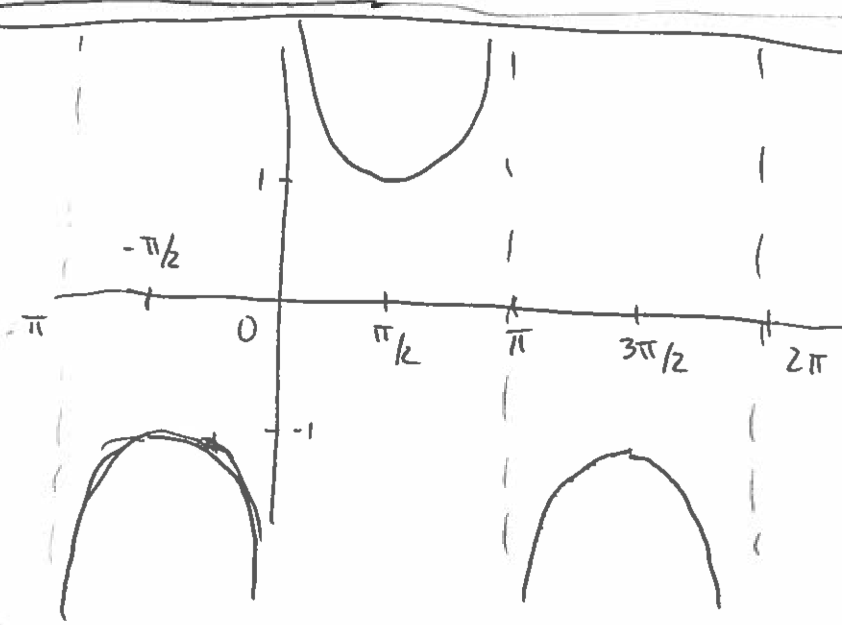
$y = \cot(x) = -\tan\left(x - \frac{\pi}{2}\right)$



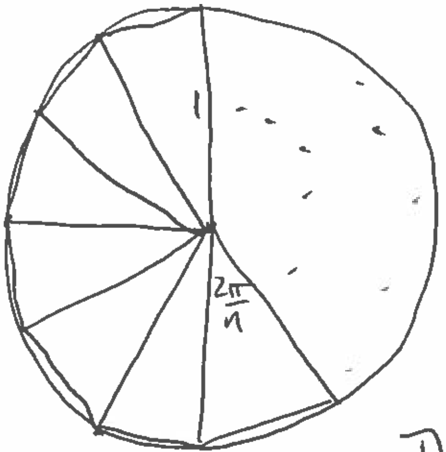
$y = \sec(x)$



$y = \csc(x) = \sec\left(x - \frac{\pi}{2}\right)$

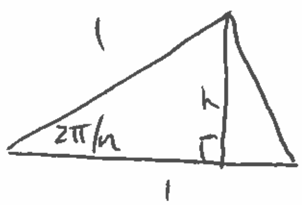


B6

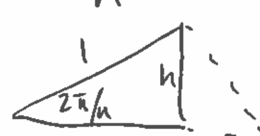


Our regular n-gon is inscribed in a circle of radius 1. From each vertex, we can draw a radius to the center of the circle. This gives n triangles, with one angle, at the vertex of the triangle at the circle's center, given by $\frac{2\pi}{n}$, since it is $\frac{1}{n}$ of the full circle. Each triangle has the same area, so if we find the area of one triangle, the area of the n-gon is n times that area. One of these triangles is pictured:

We can redraw it with one of the sides of length 1 as the base:



Now, the height is opposite the angle of measure $\frac{2\pi}{n}$ in a right triangle of hypotenuse 1:



So $h = \sin\left(\frac{2\pi}{n}\right)$. Now the area of the triangle is $\frac{1}{2}bh = \frac{1}{2}(1)\sin\left(\frac{2\pi}{n}\right) = \frac{1}{2}\sin\left(\frac{2\pi}{n}\right)$. Now the area of the n-gon is n times this, so $\boxed{\frac{n}{2}\sin\left(\frac{2\pi}{n}\right)}$