Math 103 - HW #5A,B

A1. We know that $\ln x$ is only defined for $x>0$. But, $|x| \geq 0$ for any x. So as long as $x \neq 0$, $\ln|x|$ is defined. This means the $\boxed{\text{domain of } h(x)=\ln|x| \text{ is all real } x \neq 0}$.

For any real number y, we know $e^y > 0$, and $\ln(e^y)=y$. That is, for any real number y, we can get y as an output by taking $x=e^y$ as an input, since then $h(x)=\ln|x| = \ln|e^y| = \ln e^y = y$. (for example, -100 is an output for the input $x=e^{-100}$, since $h(e^{-100}) = \ln|e^{-100}| = \ln(e^{-100}) = -100$).

$\boxed{\text{So the range is the set } \mathbb{R} \text{ of all real numbers.}}$

A2. We first note that x^2 and e^x are both defined for all real values of x, so e^{x^2} is defined for all real values of x. $\boxed{\text{So the domain of } f(x)=e^{x^2} \text{ is the set } \mathbb{R} \text{ of all real numbers.}}$

For the range, we first note that $e^x > 0$ for

A2 (cont'd) any real value x. Also, if y is any positive real number, we can get it as an output of e^x with the input $x = \ln y$, since $e^{\ln y} = y$. However, for e^{x^2}, we know $x^2 \geq 0$, so the smallest exponent we can raise e to in e^{x^2} is if $x^2 = 0$ (so $x = 0$). For $x = 0$, $e^{x^2} = e^0 = 1$, and so if $x \neq 0$, then $x^2 > 0$, so $e^{x^2} > 1$.

From this, it seems the output can be any real number $y \geq 1$. Given $y \geq 1$, consider the fact that $\ln y \geq 0$ (since $\ln y = 0$ when $y = 1$), so for the input $x = \sqrt{\ln y}$, $f(x) = e^{x^2} = e^{(\sqrt{\ln y})^2} = e^{\ln y} = y$. This means the range is the set of all real numbers $y \geq 1$.

A3. If the domain of $g(x)$ is all $x > 0$, then for $x \leq 0$, $g(x)$ is undefined. However, if $x \neq 0$, then $|x| > 0$. So $g(|x|)$ is defined if $x \neq 0$, while $g(|0|) = g(0)$ is undefined. So, the domain of $g(|x|)$ is all real numbers $x \neq 0$

A3 (cont'd) Note that an example of this is exactly as in Problem A1, where $g(x) = \ln x$ has domain $x > 0$ and $h(x) = g(|x|) = \ln|x|$ has domain $x \neq 0$.

The case $g(x^2)$ is similar, since $x^2 \geq 0$ for all x, and $x^2 > 0$ if $x \neq 0$. So if $x \neq 0$, $g(x^2)$ is defined, while $g(0^2) = g(0)$ is undefined. So, $\boxed{\text{the domain of } g(x^2) \text{ is all real } x \neq 0.}$

A4 (a) $f(g(x)) = \sqrt{g(x)} = \sqrt{x^2} = \boxed{|x|}$ (since $\sqrt{x^2} = x$ if $x \geq 0$, while $\sqrt{(-2)^2} = \sqrt{4} = 2 = -(-2) = |-2|$, so if $x < 0$, $\sqrt{x^2} = -x = |x|$).

(b) $g(f(x)) = f(x)^2 = (\sqrt{x})^2 = \boxed{x}$

(Note that the domain of $f(x)$ is all $x \geq 0$, so $(\sqrt{x})^2 = x$ for all of these x).

(c) $f(f(x)) = \sqrt{\sqrt{x}} = (x^{1/2})^{1/2} = \boxed{x^{1/4} = \sqrt[4]{x}}$

(Note that the domain of $f(x)$ and $f(f(x))$ is the set of all real $x \geq 0$).

A5: (a) $f(g(x)) = f(3^x) = (3^x)^2 + 3^x + 1 = \boxed{3^{2x} + 3^x + 1}$

(b) $g(f(x)) = 3^{f(x)} = \boxed{3^{x^2+x+1}}$

(c) $f(f(x)) = f(x^2+x+1) = (x^2+x+1)^2 + x^2+x+1 + 1 =$

$= x^4 + 2x^3 + 3x^2 + 2x + 1 + x^2 + x + 2$

$= \boxed{x^4 + 2x^3 + 4x^2 + 3x + 3}$

A6: (a) $f(g(x)) = f(\sqrt{x}) = \boxed{e^{\sqrt{x}} = e^{x^{1/2}}}$

(b) $g(f(x)) = g(e^x) = \sqrt{e^x} = (e^x)^{1/2} = \boxed{e^{x/2}}$

So if $h(x) = \frac{x}{2}$, $\boxed{f(h(x)) = f(\frac{x}{2}) = e^{x/2} = g(f(x))}$

(c) $f(f(x)) = f(e^x) = \boxed{e^{e^x}}$.

A7. (a) The x-intercepts occur when $f(x) = 0$, so $\ln(x^2-8x+13) = 0$, so $e^{\ln(x^2-8x+13)} = e^0$, or

$x^2-8x+13 = 1$, so $x^2-8x+12 = 0$. Now

$x^2-8x+12 = (x-6)(x-2) = 0$ when $x=6$ or $x=2$.

So the x-intercepts are $\boxed{x=6 \text{ and } x=2}$. The y-intercept is $f(0) = \ln(0^2-8(0)+13) = \boxed{\ln(13) = y}$

A7 (b) The x-intercepts occur when $g(x)=0$, so when $g(x)=e^{6x}-3e^{3x}-10=(e^{3x})^2-3e^{3x}-10=0$. If we think of $e^{3x}=w$, then $w^2-3w-10=0$ factors as $(w-5)(w+2)=0$, which has solutions $w=5$ and $w=-2$. With $w=e^{3x}$, this means $e^{3x}=5$, so $\ln(e^{3x})=\ln(5)$, so $3x=\ln(5)$, so $x=\frac{1}{3}\ln(5)$. Then $e^{3x}=-2$, however for any x we know $e^{3x}>0$, so $e^{3x}=-2$ is impossible (also $\ln(-2)$ is undefined). This means the only x-intercept is $\boxed{x=\frac{1}{3}\ln(5)}$.

The y-intercept is given by $g(0)=y=e^{6\cdot 0}-3e^{3\cdot 0}-10=$
$=1-3-10=\boxed{-12}$.

A8 (a) $f(x)=x^2+5x$, so $f(x+h)=(x+h)^2+5(x+h)=$
$=x^2+2xh+h^2+5x+5h$. So

$$\frac{f(x+h)-f(x)}{h}=\frac{x^2+2xh+h^2+5x+5h-(x^2+5x)}{h}=\frac{2xh+h^2+5h}{h}$$

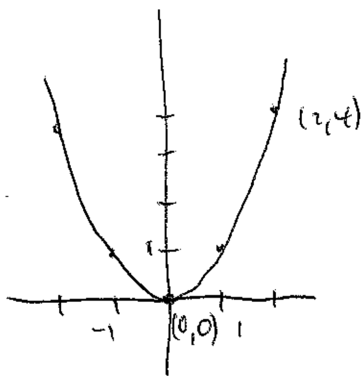
$$=\frac{h(2x+h+5)}{h}=\boxed{2x+h+5}$$

A8 (b): $f(x) = \frac{1}{x}$, so $f(x+h) = \frac{1}{x+h}$. Then

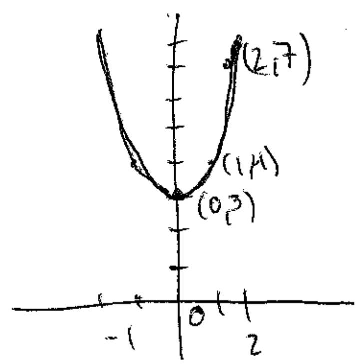
$$\frac{f(x+h) - f(x)}{h} = \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \frac{\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{h} = \frac{\frac{x-(x+h)}{x(x+h)}}{h}$$

$$= \frac{\frac{-h}{x(x+h)}}{h} = \frac{-h}{hx(x+h)} = \boxed{\frac{-1}{x(x+h)}}$$

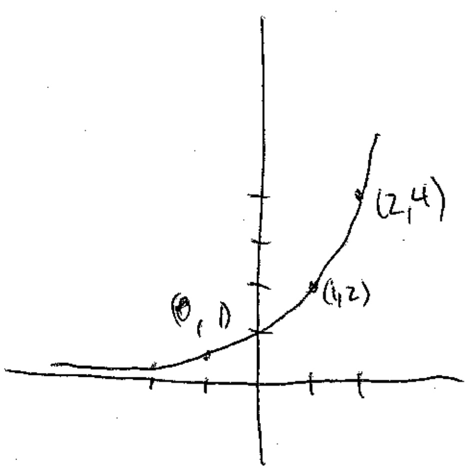
B1 (a) $y = x^2$



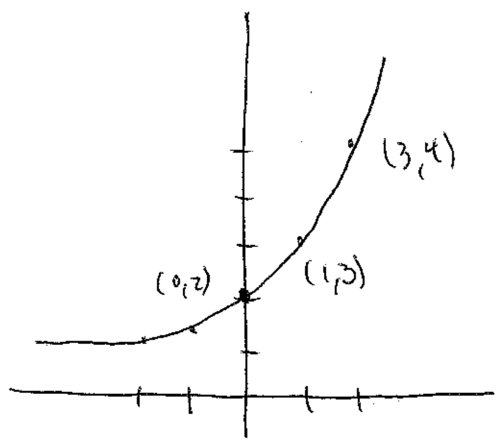
$y = x^2 + 3$



$y = 2^x$

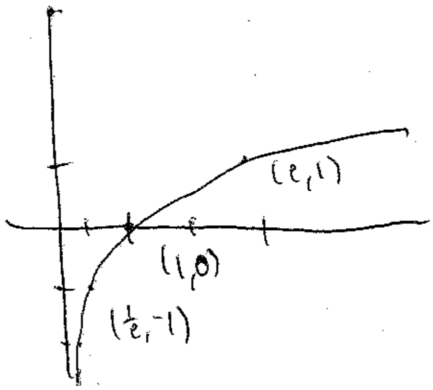


$y = 2^x + 1$

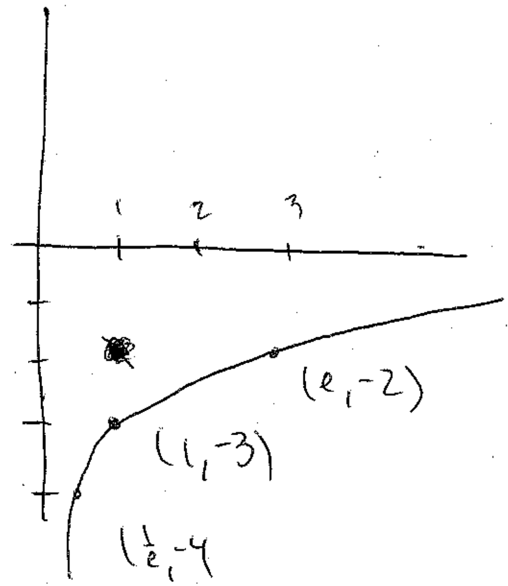


B1 (a) (cont'd)

$y = \ln x$

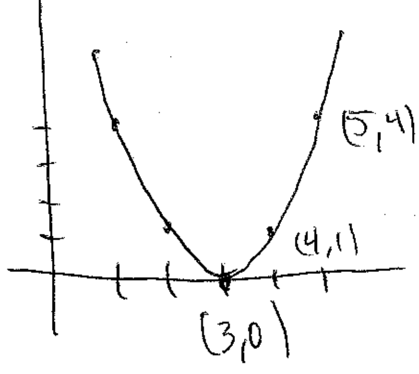


$y = \ln(x) - 3$

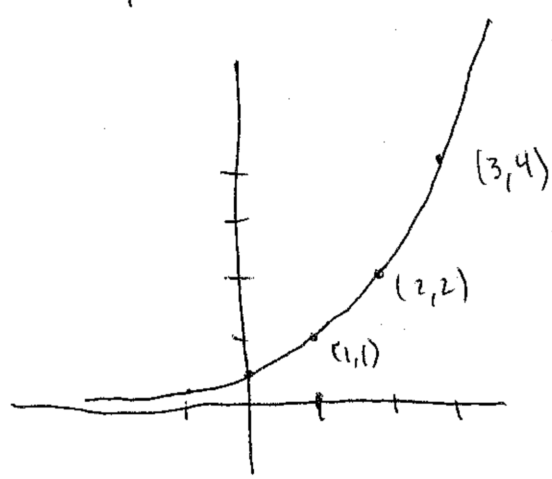


(b) If $c > 0$, the graph of $y = f(x) + c$ is the graph of $y = f(x)$ shifted up c units.
If $c < 0$, the graph of $y = f(x) + c$ is the graph of $y = f(x)$ shifted down $|c|$ units.

B2. (a) $y = (x-3)^2$

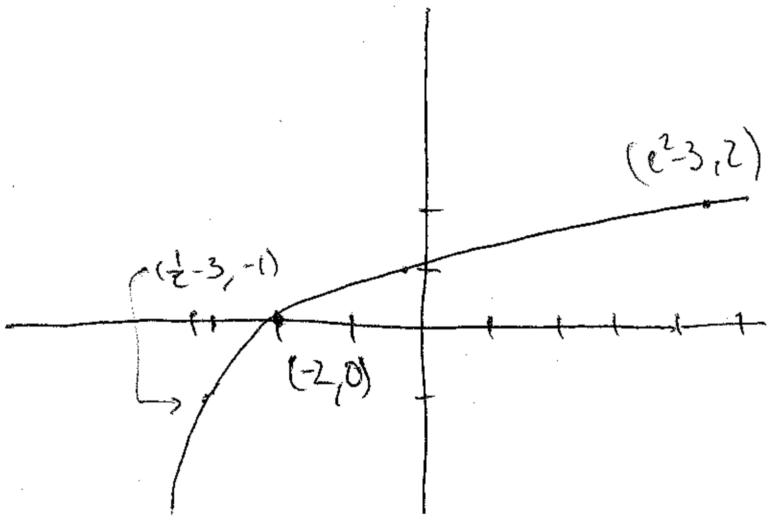


$y = 2^{x-1}$



B2 (a) (cont'd)

$y = \ln(x+3)$



x	$\ln(x+3) = y$
-2	$\ln(1) = 0$
$e-3$	$\ln(e) = 1$
$\frac{1}{e}-3$	$\ln(\frac{1}{e}) = -1$
e^2-3	$\ln(e^2) = 2$

(b) If $c > 0$, the graph of $y = f(x-c)$ is the graph of $y = f(x)$ shifted to the right by c units. The graph of $y = f(x+c)$ is the graph of $y = f(x)$ shifted to the left.

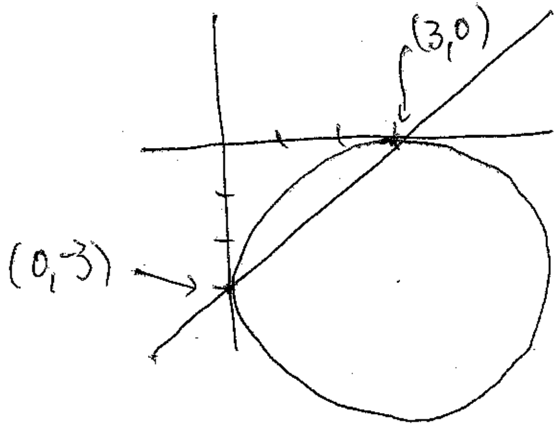
B3. (a) $y = x - 3$ and $(x-3)^2 + (y+3)^2 = 9 = 3^2$, so substituting $y = x-3$ into the second equation, we have $(x-3)^2 + (x-3+3)^2 = 9$, so

$$(x-3)^2 + x^2 = 9, \text{ so}$$

$$x^2 - 6x + 9 + x^2 = 9, \text{ so } 2x^2 - 6x = 0.$$

Then $2x(x-3) = 0$, so $x = 0$ or $x = 3$. Plugging these into $y = x - 3$ give $y = 0 - 3 = -3$, $y = 3 - 3 = 0$.

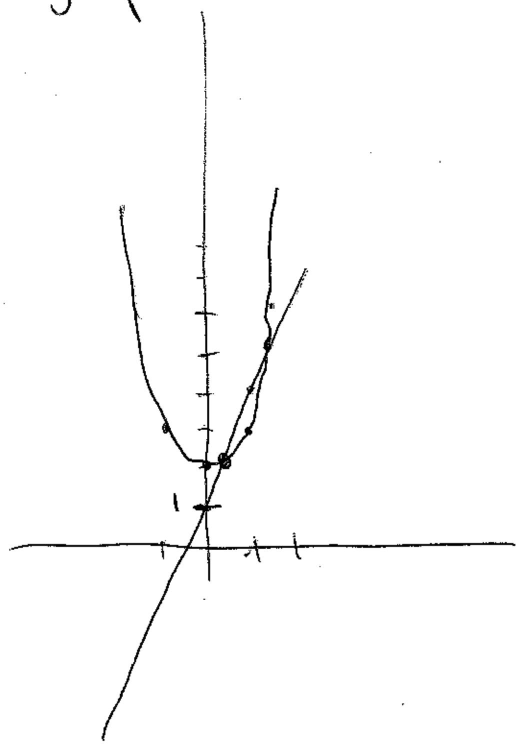
B3 (a) So the points of intersection are $(0,-3)$ and $(3,0)$. The graph of $y=x-3$ is a line with y-intercept -3 and slope 1, and the graph of $(x-3)^2+(y+3)^2=9=3^2$ is a circle with center $(3,-3)$ and radius 3. The graphs of these, with their intersection points, are:



(b) $y=x^2+2$ and $y=3x+1$ intersect when $x^2+2=3x+1$, so $x^2-3x+1=0$. From the quadratic formula, $x=\dfrac{3\pm\sqrt{9-4}}{2}=\dfrac{3\pm\sqrt{5}}{2}$. The y-values for the points of intersection can be found by plugging these into either equation, but $y=3x+1$ is a bit easier. For $x=\dfrac{3+\sqrt{5}}{2}$, $y=3\left(\dfrac{3+\sqrt{5}}{2}\right)+1=\dfrac{9+3\sqrt{5}}{2}+\dfrac{2}{2}$
$=\dfrac{11+3\sqrt{5}}{2}$, and for $x=\dfrac{3-\sqrt{5}}{2}$, $y=3\left(\dfrac{3-\sqrt{5}}{2}\right)+1$
$=\dfrac{11-3\sqrt{5}}{2}$. So the points of intersection are

B3 (b) (cont'd) $\left(\frac{3+\sqrt{5}}{2}, \frac{11+3\sqrt{5}}{2}\right)$ and $\left(\frac{3-\sqrt{5}}{2}, \frac{11-3\sqrt{5}}{2}\right)$.

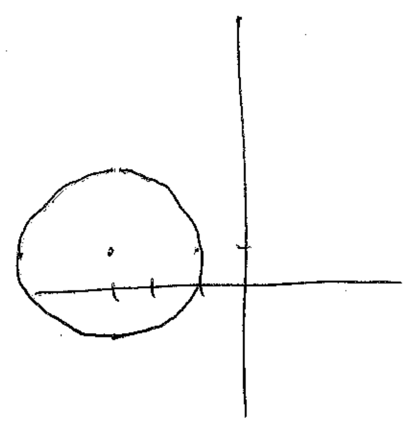
The graphs are as follows:



B4. The equation of the circle with radius r and center (a,b) is $(x-a)^2+(y-b)^2=r$. So for center $(-3,1)$ and radius 2, the equation is

$(x-(-3))^2+(y-1)^2=2^2$, or $\boxed{(x+3)^2+(y-1)^2=4}$.

The graph is given by:



B4 (cont'd) To find the functions whose graphs together give the graph of the circle, we find y in terms of x.

$$(x+3)^2+(y-1)^2=4, \text{ so } (y-1)^2=4-(x+3)^2.$$

Taking the square root is when we get two functions: $y-1=\pm\sqrt{4-(x+3)^2}$, so $y=1\pm\sqrt{4-(x+3)^2}$.

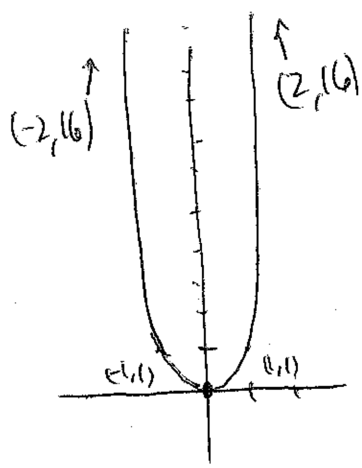
The two functions are then $\boxed{f(x)=1+\sqrt{4-(x+3)^2}}$ and $\boxed{g(x)=1-\sqrt{4-(x+3)^2}}$

B5. (a) For $f(x)=x^4$, $f(-x)=(-x)^4=(-1)^4x^4=x^4$.

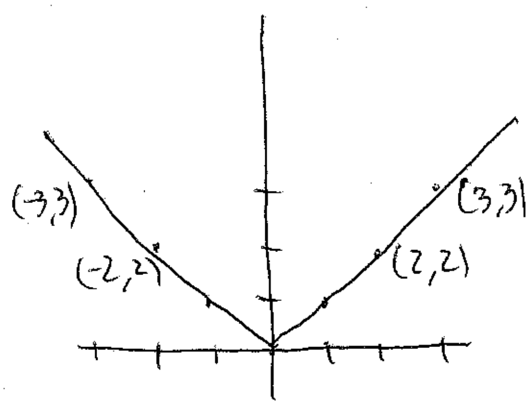
For $f(x)=|x|$, $f(-x)=|-x|=|x|$.

For $f(x)=\frac{1}{x^2}$, $f(-x)=\frac{1}{(-x)^2}=\frac{1}{(-1)^2x^2}=\frac{1}{x^2}$.

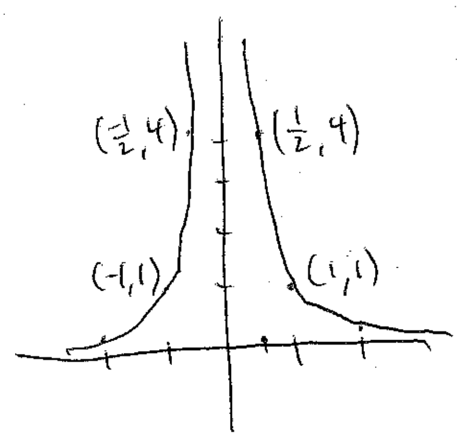
For each, $f(x)=f(-x)$. The graphs are:



$y=x^4$



$y=|x|$



$y=\frac{1}{x^2}$

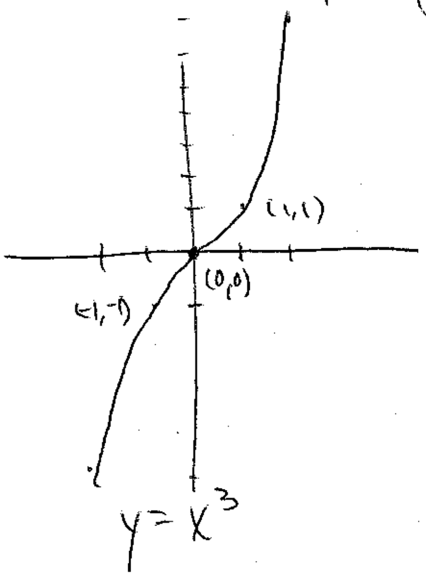
B5 (b) (cont'd)

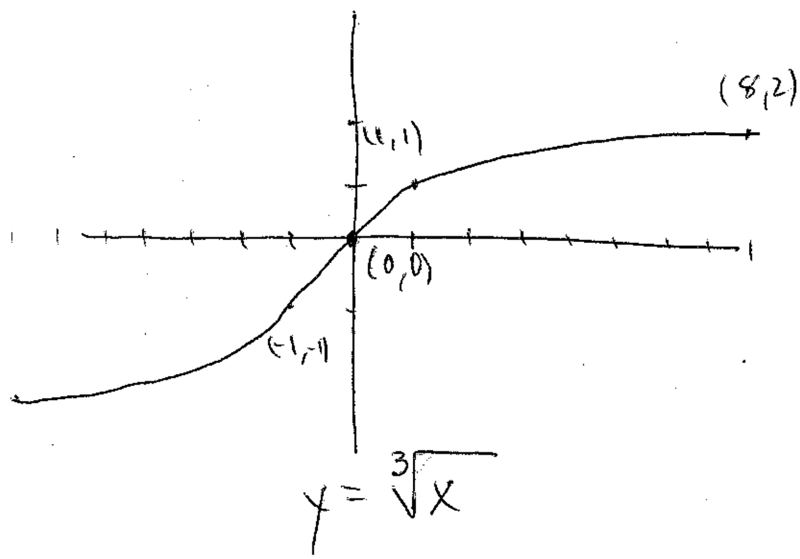
For $g(x)=x^3$, $g(-x)=(-x)^3=(-1)^3 x^3 = -x^3 = -g(x)$

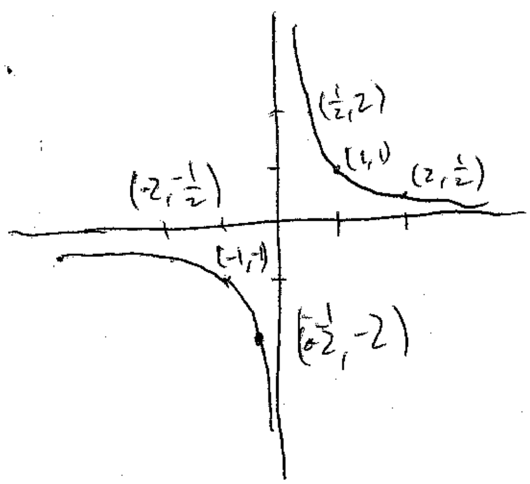
For $g(x)=\sqrt[3]{x}$, $g(-x)=\sqrt[3]{-x} = (-1)^{1/3} x^{1/3} = -1 \cdot x^{1/3} = -\sqrt[3]{x}$
$= -g(x)$

For $g(x)=\frac{1}{x}$, $g(-x)=\frac{1}{-x} = -\frac{1}{x} = -g(x)$.

So for each, $g(-x)=-g(x)$. The graphs are:

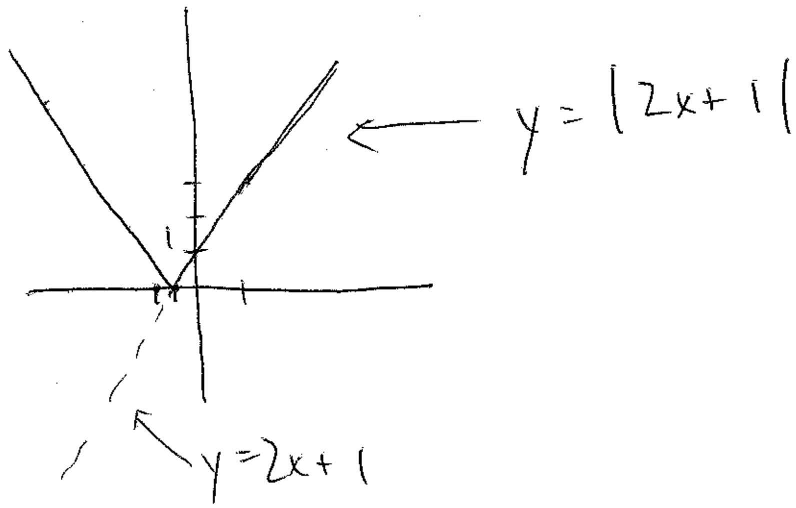






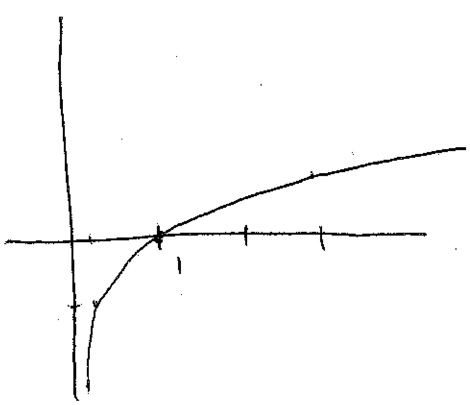
(c) For even functions, the graph is symmetric in the y-axis (mirror reflection in y-axis)
For odd functions, the graph when reflected in the x-axis then the y-axis, there is symmetry (symmetry in the origin).

B6. (a) To graph $y = |2x+1|$, first graph $y = 2x+1$, then take the part below the x-axis and reflect it to above the x-axis (negative outputs become positive)

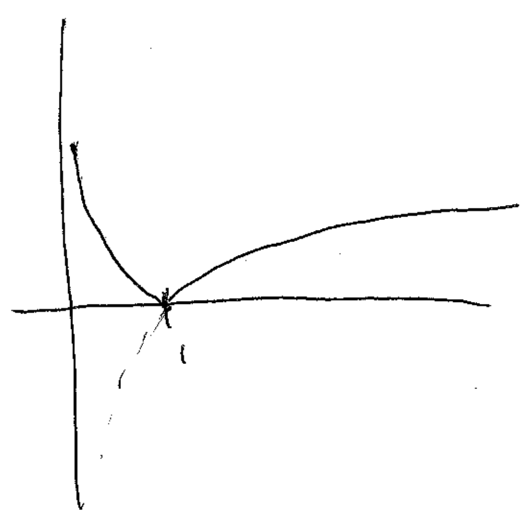


(b) Similarly, for $y = |\ln(x)|$, graph $y = \ln(x)$, and for the part of the graph below the x-axis (negative outputs), reflect to above the x-axis (positive output)

$y = \ln(x)$:

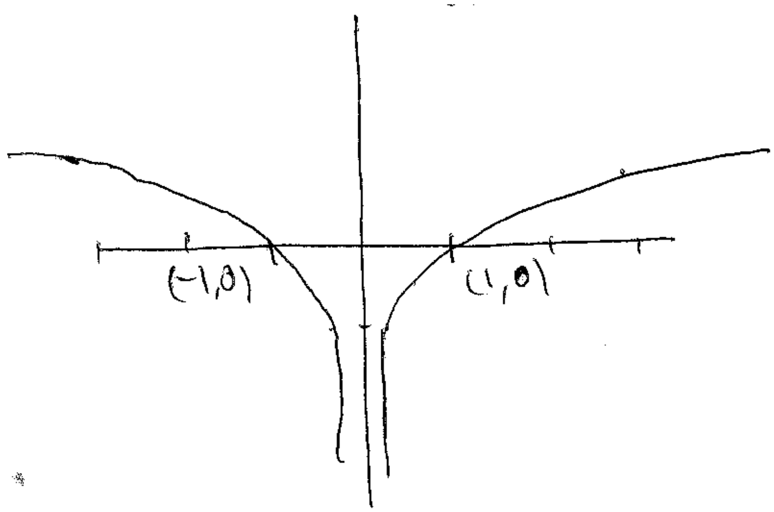


$y = |\ln(x)|$:



B6 (c) For $y = \ln|x|$, for $x > 0$ the graph is the same as $y = \ln x$. For $x < 0$, the graph of $y = \ln|x|$ looks like $y = \ln(x)$ reflected in the y-axis ($\ln|-x| = \ln|x|$, so the function is even, but not defined for $x = 0$)

$y = \ln|x|$



(d) For $|x| + |y| = 1$, plot some points. Whenever (x,y) is on the graph, so is $(\pm x, \pm y)$: