Review of Multi-variable calculus:

The functions in all models depend on two variables: time t and spatial variable x, (x, y) or (x, y, z).

The spatial variable represents the environment where the species is living (bacteria:tank in lab, rabbits and foxes:woods, birds: the space).

The time variable is one dimension, we call it **time interval**. Usually it is $(-\infty, \infty)$, $[0, \infty)$ or $[0, T]$.

In mathematics we call the environment **spatial domain** (or simply **domain**) , or **region**.

Domains

The choice of domain in a model depends on the nature of the problem.

Most of time, domain is bounded. (lab tank, woods, island, earth, universe?). And it has a boundary.

Mathematically we assume that a bounded domain is an interval (a, b) in 1-d, the region enclosed by a circular curve in 2-d, or the region enclosed by a spherical surface in 3-d.

Sometime for simplicity, or to observe certain phenomenon clearer, we also consider the whole space $\mathbf{R} = (-\infty, \infty)$, $\mathbf{R}^2$ or $\mathbf{R}^3$.

We will call a domain Ω.

Functions

Functions in the models are defined for (time interval $\times$ domain).

Let X be x, (x, y) or (x, y, z). Then the function is in a form of $f(t, X)$.

Example: Let D be a 2-d domain. (a woods)
$R(t, x, y)$ =the density of rabbit population at location (x, y) and time t
$F(t, x, y)$ =the density of fox population at location (x, y) and time t

$$\text{Population density} = \frac{\text{total population in an area}}{\text{area}}$$

Example: population density is $50,000$ per square kilometer in NYC, and it is $5,000$ in Williamsburg

Graph of the function: (hard to draw in 2-d or 3-d)

graph: $(x, y, f(x, y))$ (Maple), $(x, y, z, f(x, y, z))$.

level curve (contour): the graph of $f(x, y) = c$. (Maple)

level surface: the graph of $f(x, y, z) = c$. (Maple)

Derivatives: partial derivatives $\dfrac{\partial f(t, x, y)}{\partial t} = f_t$, $\dfrac{\partial f(t, x, y)}{\partial x} = f_x$

Gradient: $\nabla f(x, y) = (\partial f/\partial x, \partial f/\partial y)$

Gradient at one point is a vector; gradient function is a vector field; gradient vector is perpendicular to the level curve

Vector field: (a vector) $F(x,y) = (f(x,y), g(x,y))$

Jacobian: (a matrix) $J = \begin{pmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{pmatrix}$

Divergent of a vector field: (a scalar)
for $F(x,y) = (f(x,y), g(x,y))$, $div(F) = f_x + g_y$

Laplacian of a function: (a scalar)
for a function $f(x,y)$, $\Delta f = div(\nabla f) = div(f_x, f_y) = f_{xx} + f_{yy}$

Hessian of a function: (a matrix)
for a function f(x,y), Jacobian of ∇f, $H = \begin{pmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{pmatrix}$

Example: $f(x,y) = x^2 + 2y^2 - 2xy$.
(1) Find ∇f; (2) Find Hessian of f; (3) Find Δf.

Different kinds of functions:

$P(t)$: function (one variable, one function)

$P(x, y)$: multi-variable function (two variables, one function)

$(P(t), Q(t))$: vector valued function (one variable, two functions)

$(P(x, y), Q(x, y))$: vector field (two variables, two functions)

Integral of functions: Ω: two-dimensional domain, boundary $\partial\Omega$ a closed curve, $X = (x, y)$

$$\int_\Omega f(x,y)dX = \iint_\Omega f(x,y)dxdy \qquad \int_\Omega 1dX = \text{area of } \Omega$$

Divergence Theorem:

Let $\vec{F}(x,y)$ be a vector field, and let $\vec{n}(x,y)$ be the unit <u>outer</u> normal vector at (x,y), a boundary point on $\partial\Omega$. Then $\int_{\partial\Omega} \vec{F}(x,y) \cdot \vec{n}(x,y)ds$ is the total flux of $\vec{F}$ over the curve $\partial\Omega$.

$$\int_{\partial\Omega} \vec{F}(x,y) \cdot \vec{n}(x,y)ds = \int_\Omega div(\vec{F}(x,y))dX.$$

1-d: $F(b) - F(a) = \int_a^b F'(x)dx$

Green's Identities:

$$\int_\Omega u\Delta v dX = \int_{\partial\Omega} u\nabla v\cdot\vec{n}ds - \int_\Omega \nabla u\cdot\nabla v dX$$

$$\int_\Omega u\Delta v dX - \int_\Omega v\Delta u dX = \int_{\partial\Omega} u\nabla v\cdot\vec{n}ds - \int_{\partial\Omega} v\nabla u\cdot\vec{n}ds$$

Example: Let $F(x,y) = (x+y, e^{x-y})$, and let Ω be a square $(0,1)\times(0,1)$.

(1) Calculate $\int_\Omega div(\vec{F}(x,y))dX$

(2) calculate $\int_{\partial\Omega} \vec{F}(x,y)\cdot\vec{n}(x,y)ds$

Differential Equations: (continuous model)

Malthus equation: $\dfrac{dN}{dt} = rN$, Solution: $N(t) = N_0 e^{rt}$

Assumption: the reproduction rate is proportional to the size of the population

Logistic equation: $\dfrac{dN}{dt} = rN\left(1 - \dfrac{N}{K}\right)$,

Solution: $N(t) = \dfrac{KN_0}{(K - N_0)e^{-rt} + N_0}$

Assumptions: the reproduction rate is proportional to the size of the population when the population size is small, and the growth is negative when the size is large

[B] Section 1.3

General ODE growth model: $\frac{dN}{dt} = Ng(N)$,
$g(N)$ is growth rate per capita

Malthus: $g(N)$ is a constant
Logistic: $g(N)$ is decreasing (compensatory, crowding effect)
Weak Allee effect: $g(N)$ is first increasing, then decreasing, and $g(0) > 0$ (depensatory)
Strong Allee effect: $g(N)$ is first increasing, then decreasing, and $g(0) < 0$ (critical depensatory)

Harvesting: $\frac{dN}{dt} = Ng(N) - h(N)$
$h(N)$ is the harvesting rate

Qualitative behavior of solutions:
the most common case is that the solution tends to an equilibrium $N(t) = C$.

Stability of an equilibrium point:

Suppose that $y = y_0$ is an equilibrium point of $y' = f(y)$.

y_0 is a **sink** if any solution with initial condition close to y_0 tends toward y_0 as t increase.

y_0 is a **source** if any solution with initial condition close to y_0 tends toward y_0 as t decrease.

y_0 is a **node** if it is neither a sink nor a source.

Linearization Theorem:

Suppose that $y = y_0$ is an equilibrium point of $y' = f(y)$.

If $f'(y_0) < 0$, then y_0 is a sink;
If $f'(y_0) > 0$, then y_0 is a source;
If $f'(y_0) = 0$, then y_0 can be any type, but in addition if $f''(y_0) > 0$ or $f''(y_0) < 0$, then y_0 is a node.

Bifurcation: Suppose that the differential equation depends on a parameter. Then we say that a bifurcation occurs if there is a qualitative change in the behavior of solutions as the parameter changes.

Types of bifurcations

Example 1: $\dfrac{dy}{dt} = ky(1-y)$ (no bifurcation)

Example 2: $\dfrac{dy}{dt} = y^2 - \mu$ (saddle-node bifurcation, supercritical)

Example 3: $\dfrac{dy}{dt} = y^3 + \mu y$ (pitchfork bifurcation, subcritical)

Example 4: $\dfrac{dy}{dt} = y^2 - \mu y$ (transcritical bifurcation)

A Harvesting Model: Holling's type II model, Michaelis-Menton kinetics in biochemistry

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right) - \frac{AP}{1 + BP}$$

Assumptions: The number of predator is assumed to be constant, and they cannot consume more preys when P is large. It takes the predator a certain amount of time to kill and eat each prey. So suppose that in one hour, the predator (a wolf) can catch AP number of prey (rabbits) (it is proportional to P since when P is larger, the wolf has better chance to meet rabbits,) but it needs T hour to handle and eat each rabbit caught. So for all AP rabbits, it takes ATP hours, and in fact the wolf spends $1 + ATP$ hours on these AP rabbits. So in 1 hour, the wolf actually only eats $\dfrac{AP}{1 + ATP}$ rabbits. We use $B = AT$ as a new parameter in the equation.

Example of analysis of the model: $\dfrac{dQ}{ds} = Q\,(1-Q) - \dfrac{hQ}{1+aQ}$

$Q\,(1-Q) - \dfrac{hQ}{1+aQ} = 0, \quad Q = 0$ or $aQ^2 + (1-a)Q + (h-1) = 0,$

$Q_{\pm} = \dfrac{a-1 \pm \sqrt{(a+1)^2 - 4ah}}{2a}$, Basic border line: $h = \dfrac{(a+1)^2}{4a}$

when $0 < h < \dfrac{(a+1)^2}{4a}$, three equilibrium points

when $h = \dfrac{(a+1)^2}{4a}$, two equilibrium points (except $a = 1$)

when $h > \dfrac{(a+1)^2}{4a}$, one equilibrium points

But we also count the negative equilibrium points

Trace-determinant analysis:

$0 = aQ^2 + (1-a)Q + (h-1) = a(Q-Q_1)(Q-Q_2)$

$Q_1 > 0, Q_2 > 0$ if $1-a < 0$ and $h-1 > 0$
$Q_1 > 0, Q_2 < 0$ if $h-1 < 0$
$Q_1 < 0, Q_2 < 0$ if $1-a > 0$ and $h-1 > 0$

Now we have a complete classification