# Once Upon a Time...

Constant diffusion:

$$J = -D\nabla \cdot u$$

$$\frac{\partial u}{\partial t} = -\nabla \cdot J = D\Delta u$$

But is this truly realistic???

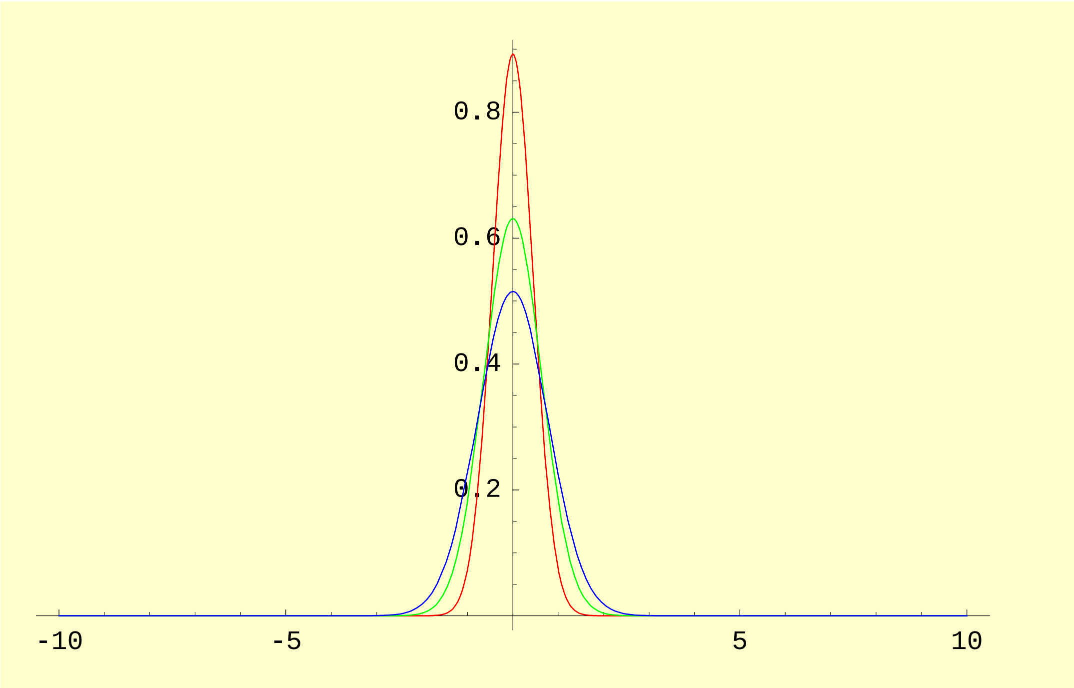
# Problems With Constant Diffusion

Any initial condition, even a point distribution, instantly "spreads out" to cover an infinite domain. Consider the one-dimensional case:

$$\begin{cases} \frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} \\ u(x, 0) = \delta(x) \end{cases}$$

$$\Rightarrow u(x, t) = \frac{1}{\sqrt{4 \pi D t}} \exp\left(- \frac{x^2}{4 D t}\right)$$

# Problems With Constant Diffusion



# How Can We Improve This?

Use a non-constant diffusion term:

$$J = -D(u)\,\nabla \cdot u(x, t)$$

$$\frac{\partial u}{\partial t} = -\nabla \cdot J = \nabla \cdot (D(u)\,\nabla \cdot u)$$

This makes intuitive sense - in an insect population, for example, we would expect very densely populated areas to diffuse outwards more quickly than sparsely populated areas.

# Crap!

Of course, now we need to figure out how to deal with non-constant diffusion in our solution.

# A General Approach

Rewrite our equation as

$$\frac{\partial u}{\partial t} - \nabla \cdot (D(u)\, \nabla \cdot u) = 0$$

We can consider this to be an example of a general class of functions of the form

$$G(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}) = 0$$

# A General Approach

The *one-parameter family of stretching functions:*

$$\overline{x} = \epsilon^{a}\, x$$

$$\overline{t} = \epsilon^{b}\, t$$

$$\overline{u} = \epsilon^{c}\, u$$

# A General Approach

The *one-parameter family of stretching functions:*

$$\overline{x} = \epsilon^a x$$

$$\overline{t} = \epsilon^b t$$

$$\overline{u} = \epsilon^c u$$

$a$, $b$, and $c$ are constants; $\epsilon$ is a real parameter on some open interval that contains 1.

Define $G$ to be *invariant* if there exists a smooth function $f(\epsilon)$ such that

$$G(\overline{x}, \overline{t}, \overline{u}, \overline{u}_{\overline{x}}, \overline{u}_{\overline{t}}, \overline{u}_{\overline{x}\overline{x}}, \overline{u}_{\overline{x}\overline{t}}, \overline{u}_{\overline{t}\overline{t}}) = f(\epsilon)\, G(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt})$$

# A General Approach

Assume $G$ is invariant. This gives us

$$G(\overline{x}, \overline{t}, \overline{u}) = f(\epsilon)\, G(x, t, u)$$

$$G(\epsilon^a x, \epsilon^b t, \epsilon^c u) = f(\epsilon)\, G(x, t, u)$$

$$G(\epsilon^a x, \epsilon^b t, \epsilon^c u) = f(\epsilon)\, (0)$$

$$G(\epsilon^a x, \epsilon^b t, \epsilon^c u) = 0$$

(Because $G$ is homogenous.)

# A General Approach

Differentiate with respect to $\epsilon$:

$$a\,x\,\epsilon^{a}\,\frac{\partial G}{\partial x} + b\,t\,\epsilon^{b}\,\frac{\partial G}{\partial t} + c\,u\,\epsilon^{c}\,\frac{\partial G}{\partial u} = 0$$

Set $\epsilon = 1$ (which we can do because we restrict $\epsilon$ to a domain that contains 1):

$$a\,x\,\frac{\partial G}{\partial x} + b\,t\,\frac{\partial G}{\partial t} + c\,u\,\frac{\partial G}{\partial u} = 0$$

Clever people look at this and see that the transformation we want to use is

$$u = t^{c/b}\,r(z)$$

$$z = \frac{x}{t^{a/b}}$$

# A General Approach

Verification of the transformation:

$$a\,x\,\frac{\partial G}{\partial x} + b\,t\,\frac{\partial G}{\partial t} + c\,u\,\frac{\partial G}{\partial u} = 0$$

$$a\,x\,\frac{\partial G}{\partial z}\,\frac{\partial z}{\partial x} + b\,t\,\frac{\partial G}{\partial z}\,\frac{\partial z}{\partial t} + c\,u\,\frac{\partial G}{\partial z}\,\frac{\partial z}{\partial u} = 0$$

$$\frac{\partial z}{\partial x} = \frac{1}{t^{a/b}},\ \frac{\partial z}{\partial t} = \cdots$$

# A General Approach

What have we accomplished with all our fancy math?

$$G(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}) = 0$$

$$\Longleftrightarrow$$

$$g(z, r, r', r'') = 0$$

# A Specific Example

Recall the problem we're actually working on:

$$\frac{\partial u}{\partial t} = \nabla \cdot (D(u) \nabla \cdot u)$$

$$D(u) = D_0 \left( \frac{u}{u_0} \right)^m$$

# A Specific Example

Letting $m = 1$ gives us

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\frac{D_0}{u_0}\, u\, \frac{\partial u}{\partial x}\right)$$

And because we're lazy, we'll assume $\frac{D_0}{u_0} = 1$, so

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u\, \frac{\partial u}{\partial x}\right)$$

The problem is now

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u\, \frac{\partial u}{\partial x}\right) \\ u(x, 0) = \delta(x) \end{cases}$$

# A Specific Example

Other assumptions:

Since no organisms are being born or dying, we require for all $t > 0$

$$\int_{-\infty}^{\infty} u(x, t)\, dx = 1$$

and

$$\lim_{x \to \pm\infty} u(x, t) = 0$$

# A Specific Example

Check for invariance:

$$G(\overline{x},\, \overline{t},\, \overline{u},\, \overline{u}_{\overline{x}},\, \overline{u}_{\overline{t}},\, \overline{u}_{\overline{x}\overline{x}},\, \overline{u}_{\overline{x}\overline{t}},\, \overline{u}_{\overline{t}\overline{t}}) = f(\epsilon)\, G(x,\, t,\, u,\, u_x,\, u_t,\, u_{xx},\, u_{xt},\, u_{tt})$$

$$\overline{x} = \epsilon^a\, x$$

$$\overline{t} = \epsilon^b\, t$$

$$\overline{u} = \epsilon^c\, u$$

$$\frac{\partial \overline{u}}{\partial \overline{t}} - \frac{\partial}{\partial \overline{x}} \left( \overline{u}\, \frac{\partial \overline{u}}{\partial \overline{x}} \right) = \epsilon^{c-b}\, \frac{\partial u}{\partial t} - \epsilon^{2\,c-2\,a}\, \frac{\partial}{\partial x} \left( u\, \frac{\partial u}{\partial x} \right)$$

We have invariance if

$$c - b = 2\,c - 2\,a \Rightarrow c = 2\,a - b$$

# A Specific Example

$$u = t^{c/b}\, r(z)$$

$$z = \frac{x}{t^{a/b}}$$

With our invariance condition,

$$u = t^{(2\,a-b)/b}\, r(z)$$

$$z = \frac{x}{t^{a/b}}$$

# A Specific Example

Let's be clever:

$$\int_{-\infty}^{\infty} u(x, t)\, dx = 1$$

$$t^{(2a-b)/b} \int_{-\infty}^{\infty} r\left(\frac{x}{t^{a/b}}\right) dx = 1$$

$$t^{(3a-b)/b} \int_{-\infty}^{\infty} r(z)\, dz = 1$$

Time-independence requires

$$b = 3a$$

Which simplifies the transformation to

$$u = t^{-1/3}\, r(z)$$

$$z = x\, t^{-1/3}$$

# A Specific Example

$$\left\{\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial}{\partial x}\left(u\,\frac{\partial u}{\partial x}\right) \\ u &= t^{-1/3}\,r\,(z) \\ z &= x\,t^{-1/3} \end{aligned}\right. \Longrightarrow 3\,(r\,r')' + r + z\,r' = 0$$

This equation can be integrated to give

$$3\,r\,r' + z\,r = \text{constant}$$

Take the constant to be zero; the solution is

$$r(z) = \frac{A^2 - z^2}{6}$$

# A Specific Example

Use our conditions to clean it up:

$$\lim_{x \to \pm\infty} u(x, t) = 0$$

means that

$$r(z) = \begin{cases} \frac{A^2 - z^2}{6}, & |x| < A \\ 0 & |x| > A \end{cases}$$

And

$$\int_{-\infty}^{\infty} u(x, t)\, dx = 1$$

means that

$$A = \left(\frac{9}{2}\right)^{1/3}$$

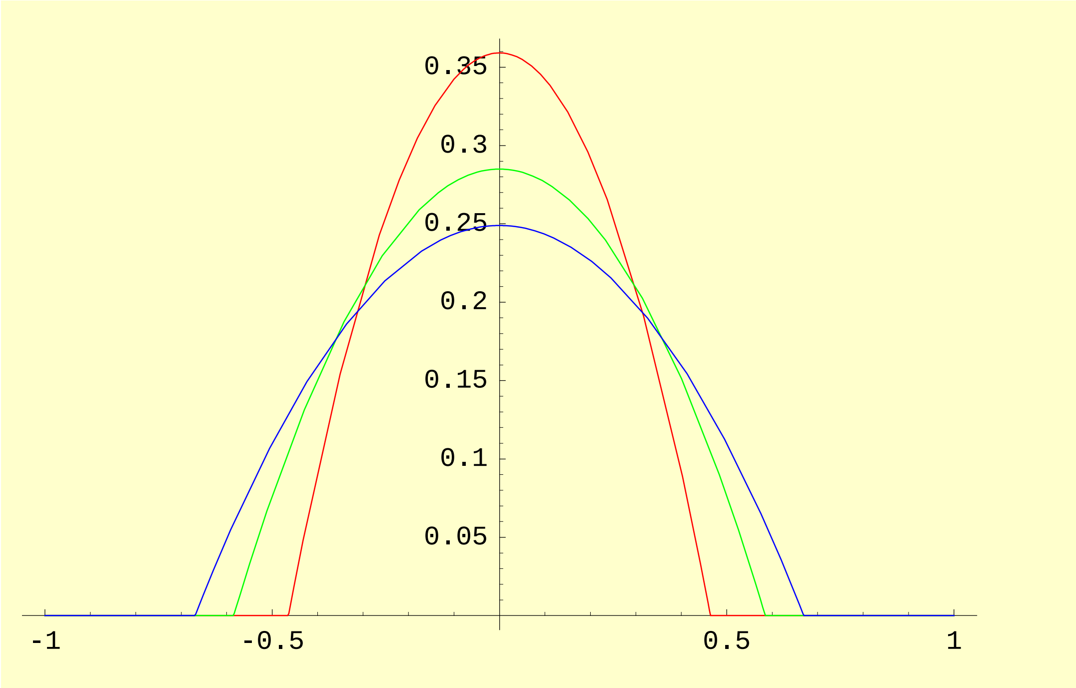
Now switch everything back to original coordinates.

# A Specific Example

$$u(x,\ t) = \begin{cases} \frac{1}{6\,t}\,(A^2\,t^{2/3} - x^2), & |x| < A\,t^{1/3} \\ 0 & |x| > A\,t^{1/3} \end{cases}$$

About time. Let's take a look!

# A Specific Example - Pretty Pictures



# A Specific Example

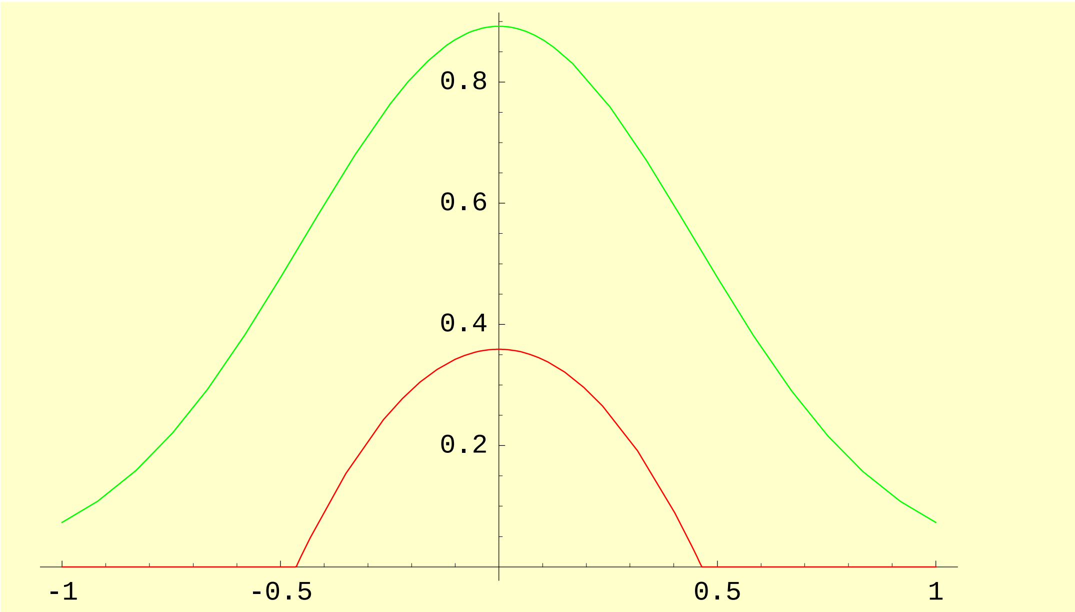
The key feature of this solution is the sharp wave front at

$$x_f = A\,t^{1/3}$$

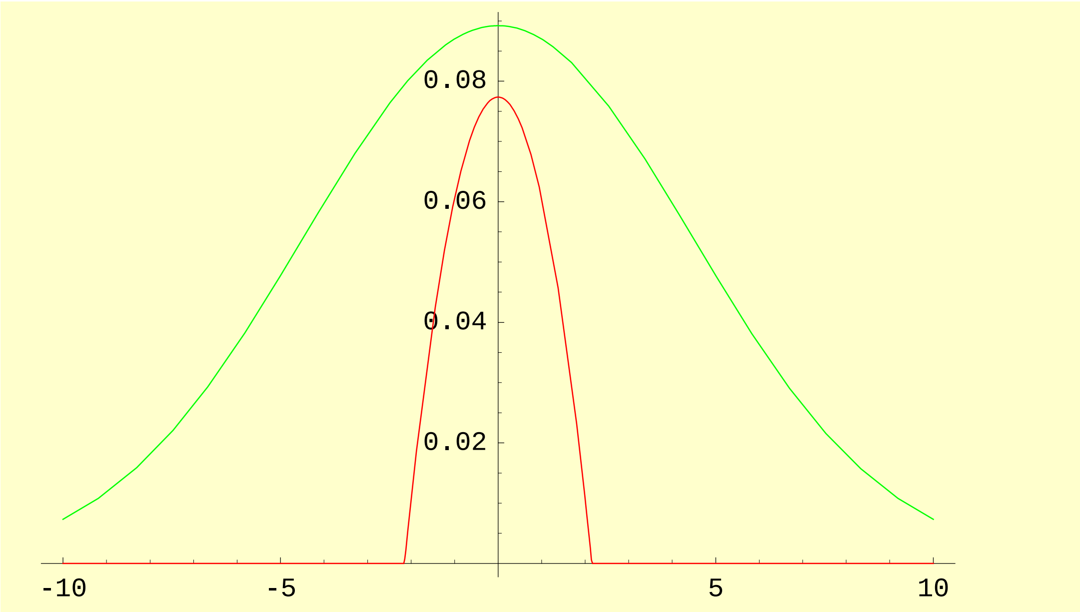
This wave is moving with speed

$$\frac{d x_f}{d t} = \frac{1}{3} A\,t^{-2/3}$$

# Comparing Constant and Density-Dependent Diffusion at *t* = .1



# Comparing Constant and Density-Dependent Diffusion at $t = 10$



# What About the Not-Simple Case, You Ask?

Recall that the general form is

$$\frac{\partial u}{\partial t} = \nabla \cdot (D(u) \nabla \cdot u)$$

$$D(u) = D_0 \left( \frac{u}{u_0} \right)^m$$

and we assumed $m = 1$ for all the work we just did. Is there a general solution?

# What About the Not-Simple Case, You Ask?

Yes, and here it is:

$$u(x,\, t) = \begin{cases} \frac{u_0}{\lambda(t)} \left(1 - \left(\frac{x}{r_0\, \lambda(t)}\right)^2\right)^{1/m}, & |x| \le r_0\, \lambda(t) \\ 0 & |x| > r_0\, \lambda(t) \end{cases}$$

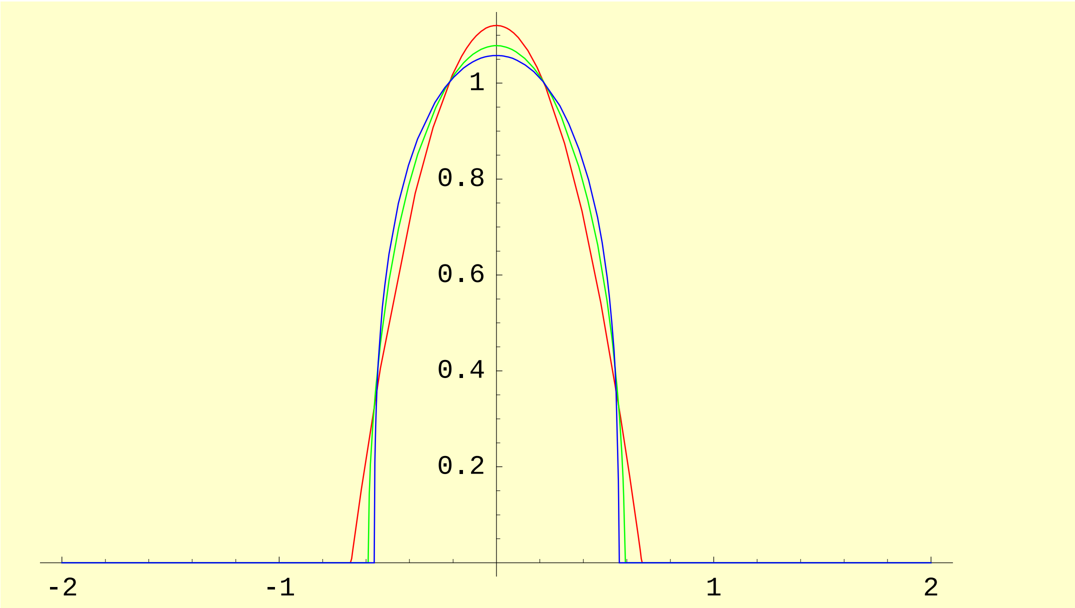
Where

$$\lambda(t) = \left(\frac{t}{t_0}\right)^{1/(2+m)}$$

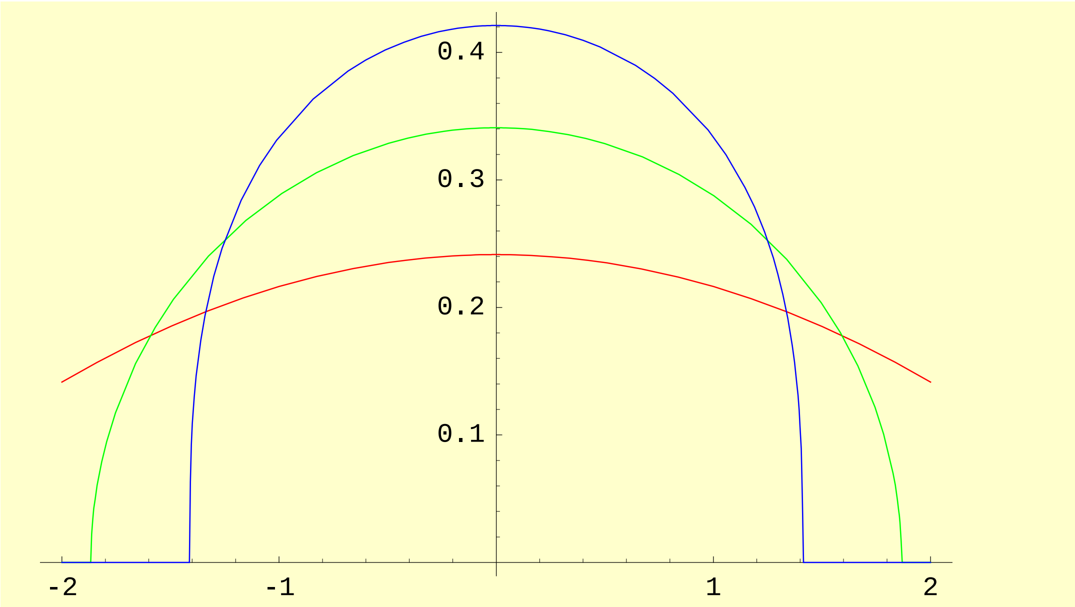
$$r_0 = \frac{Q\, \Gamma\left(\frac{1}{m} + \frac{3}{2}\right)}{\pi^{1/2}\, u_0\, \Gamma\left(\frac{1}{m} + 1\right)}, \quad t_0 = \frac{r_0^2\, m}{2\, D_0 (m+2)}$$

$D_0$ and $n_0$ are positive constants; $Q$ is the initial density at the origin, and $r_0$ comes from requiring that the integral over the domain at all times be equal to $Q$.

# Not-Simple Case, *t* = .1 and *m* = 1, 2, 3



# Not-Simple Case, $t = 10$ and $m = 1, 2, 3$



# Pay No Attention to the Man Behind the Curtain

In[30]:=

$$u[x\_, t\_] := \text{Which}\left[\text{Abs}[x] < t^{1/3}, \frac{1}{6\,t}\left(t^{2/3} - x^2\right), \text{True}, 0\right]$$

In[31]:=

$$\text{uConstantD}[x\_, t\_] := \frac{1}{\sqrt{4\,\pi\,t}}\,\text{Exp}\left[\frac{-x^2}{4\,t}\right]$$

In[39]:=

```
Plot[{u[x, .1], u[x, .2], u[x, .3]},
  {x, -1, 1}, PlotStyle → {RGBColor[1, 0, 0],
    RGBColor[0, 1, 0], RGBColor[0, 0, 1]}];
```

In[40]:=

```
Plot[{u[x, .1], uConstantD[x, .1]},
  {x, -1, 1}, PlotStyle → {RGBColor[1, 0, 0],
    RGBColor[0, 1, 0], RGBColor[0, 0, 1]}];
```

In[46]:=
```
Plot[{u[x, 10], uConstantD[x, 10]},
  {x, -10, 10}, PlotRange → All,
  PlotStyle → {RGBColor[1, 0, 0],
    RGBColor[0, 1, 0], RGBColor[0, 0, 1]}];
```

In[47]:= $\lambda[t\_, m\_] := \left(\frac{t}{t0[m]}\right)^{\frac{1}{2+m}}$

In[48]:= $r0[m\_] := \frac{\text{Gamma}\left[\frac{1}{m} + 3/2\right]}{\sqrt{\pi}\ \text{Gamma}[1/m + 1]}$

In[49]:= $t0[m\_] := \frac{r0[m]^2 * m}{2\,(m+1)}$

In[50]:= $\text{uGeneral}[x\_, t\_, m\_] :=$
$\text{Which}\Big[\text{Abs}[x] \le r0[m]\,\lambda[t, m],$
$\frac{1}{\lambda[t, m]}\left(1 - \left(\frac{x}{r0[m]\,\lambda[t, m]}\right)^2\right)^{1/m}, \text{True}, 0\Big]$

In[54]:=
```
Plot[{uGeneral[x, .1, 1],
    uGeneral[x, .1, 2], uGeneral[x, .1, 3]},
  {x, -2, 2}, PlotRange → All,
  PlotStyle → {RGBColor[1, 0, 0],
     RGBColor[0, 1, 0], RGBColor[0, 0, 1]}];
```

In[55]:=
```
Plot[{uGeneral[x, 10, 1],
    uGeneral[x, 10, 2], uGeneral[x, 10, 3]},
  {x, -2, 2}, PlotRange → All,
  PlotStyle → {RGBColor[1, 0, 0],
     RGBColor[0, 1, 0], RGBColor[0, 0, 1]}];
```