Midterm exercise

Math 490

Due: March 20th, 5pm

1. Consider a diffusion equation with convection

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + 4\dfrac{\partial u}{\partial x} + 8u, \quad t > 0,\ x \in (0, L), \\ u(t, 0) = 0,\ u(t, L) = 0, \\ u(0, x) = u_0(x), \quad x \in (0, L). \end{cases} \tag{1}$$

   Find the solution of the equation with the following steps:

   (a) Use separation of variables method to show that if $u(t, x) = U(t)V(x)$ is a solution, then for some constant k, U and V satisfy

   $$U'(t) = kU(t), \quad V''(x) + 4V'(x) + 8V(x) = kV(x), \quad V(0) = V(L) = 0.$$

   (b) Find the eigenvalues and eigenfunctions of

   $$V''(x) + 4V'(x) + 8V(x) = kV(x), \quad V(0) = V(L) = 0.$$

   (Hint: treat the cases of $k > 4$, $k = 4$ and $k < 4$ separately.)

   (c) Find the solution of the equation in a series form. (Hint: $c_n = \dfrac{2}{L}\displaystyle\int_0^L e^{2x}u_0(x)\sin\left(\dfrac{n\pi x}{L}\right)dx$.)

   (d) Determine the critical patch L_0 of the problem, and describe the population distribution qualitatively when $L > L_0$.

2. Find the general solution of the equation: $u_{xx} + xu_x + u = 0$, $x \in (-\infty, \infty)$, and the one tends to zero as $|x| \to \infty$. (Hint: product rule of differentiation: $(xu)_x = xu_x + u$)

3. Turchin (Derek's presentation) suggested a nonlinear diffusion equation in the form of

   $$u_t = \left(au - bu^2 + u^3\right)_{xx}, \quad x \in (-\infty, \infty), \quad t > 0.$$

   We assume $a = b = 3$ here. Find an exact solution of the problem with the following steps:

   (a) Use change of variables $u - 1 = v$ to rewrite the equation to the form $v_t = (v^3)_{xx}$.

   (b) Find a solution of $v_t = (v^3)_{xx}$ with the self-similar form $v(x, t) = t^a w(t^b x)$ by solving the ordinary differential equation satisfied by w (Daniel's presentation). We assume the solution v is positive in an interval $[-L, L]$, and is zero otherwise

   (c) Write the exact form of solution $u(x, t)$.

4. (Spread of gypsy moths) Gypsy moths (*Lymantria dispar*) were brought to Massachusetts from Europe around 1870 in connection with silkworm development research. Needless to say, some of the moths escaped from breeding cages but somehow large growths and widespread dispersal were kept under control for a number of years. However, around 1900 there was a drastic increase of gypsy moth population in the Boston area which quickly spread to adjacent regions. By 1925 or so, when dispersal was finally halted, gypsy moths covered all of New England and parts of New York state and Canada. There was severe damage to

forests throughout the region. The following are the cumulative areas corresponding to the dispersion fronts. According to the studies by Elton, there was no significant expansion of the front after 1925.

Year	1900	1905	1910	1915	1920	1925
Area (km^2)	1290	9080	26960	58840	79770	113320

Use the data above to estimate the value of aD and the year when the area was zero in this example. (Hint: modify the `Maple` program for muskrat population, use function $A(x - B)^2$ instead of Ax^2 since our first area data is not zero, and notice that Boston is a coastal city, so the spread areas are semicircular instead of circular.)