Chapter 3

Diffusion of point source and biological dispersal

3.1 Diffusion of a point source

Suppose that a new species is introduced to an existing environment, then naturally the individuals of the species will disperse from the release point. That would similar to the diffusion when a drop of ink enter a jar of pure water. Reaction-diffusion equation can be used to describe an ideal growth and spatial-diffusion phenomena. The simplest way is to consider the dispersion in an infinite domain–we first consider the simplest one dimensional problem. Suppose that $P(t,x)$ is the population density function of this species, $t \geq 0$ and $x \in (-\infty, \infty)$. Suppose that at $t = 0$, we import a group of M individuals to a point (say, $x = 0$). Let $u_0(x)$ be the initial distribution of the population. Then

$$\int_{-\infty}^{\infty} u_0(x)dx = M. \tag{3.1}$$

However $u_0(x) = 0$ for any $x \neq 0$ since before the dispersion, the species does not exist in the environment yet. On the other hand the population density at $x = 0$ is given by the following limit

$$u_0(0) = \lim_{x \to 0^+} \frac{\int_{-x}^{x} u_0(x)dx}{2x} = \infty. \tag{3.2}$$

Thus $u_0(x)$ is an unusual function can be characterized by

$$u_0(x) = \begin{cases} \infty & \text{if } x = 0, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} u_0(x)dx = M. \tag{3.3}$$

This function is called a δ-function (delta function) in physics, and we denote it by $\delta_0(x)$ (a function which is infinity at a would be $\delta_a(x)$.) Although this function is not a function in the normal sense, we can check that the function

$$\Phi(t,x) = \frac{M}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}} \tag{3.4}$$

is a solution of

$$u_t = Du_{xx}, \ \ t > 0, \ x \in \mathbf{R}, \ \ u(0, x) = M\delta_0(x), \tag{3.5}$$

in the sense that Φ satisfies the equation when $t > 0$ and $x \in \mathbf{R}$, and

$$\lim_{t \to 0^+} \Phi(t, x) = 0, \ \ x \neq 0, \ \ \lim_{t \to 0^+} \Phi(t, 0) = \infty. \tag{3.6}$$

The function Φ is called the *fundamental solution* of one-dimensional diffusion equation. For each fixed $t > 0$, $\Phi(t, \cdot)$ is a normal distribution function, thus the graph of $\phi(t, \cdot)$ is a bell-shaped curve. Moreover since the individuals in the population only move around in the environment, there is no new reproduction, the total population keeps constant:

$$\int_{-\infty}^{\infty} \Phi(t, x) dx = M. \tag{3.7}$$

Biologically, dispersion in a two-dimensional space is more practical. Similar to one-dimensional case, the equation

$$u_t = D(u_{xx} + u_{yy}), \ \ t > 0, \ (x, y) \in \mathbf{R}^2, \ \ u(0, x, y) = \delta_{(0,0)}(x, y), \tag{3.8}$$

has solution

$$\Phi(t, x, y) = \frac{M}{4\pi Dt} e^{-\frac{x^2+y^2}{4Dt}}. \tag{3.9}$$

Here $\delta_{(0,0)}(x, y)$ is a function defined as

$$\delta_{(0,0)}(x, y) = \begin{cases} \infty & \text{if } (x, y) = (0, 0), \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \int_{\mathbf{R}^2} \delta_{(0,0)}(x, y) dx dy = M. \tag{3.10}$$

We also notice that $\Phi(t, x, y)$ is radially symmetric, thus the diffusion from a point source in two-dimensional space is radial, which can be regarded as a consequence of the rotational symmetry of the Laplacian operator.

3.2 Mathematical derivation of the fundamental solutions

In this section, we will solve the equation

$$u_t = Du_{xx}, \ \ t > 0, \ x \in \mathbf{R}, \ \ u(0, x) = \delta_0(x), \tag{3.11}$$

and derive the formula of the fundamental solution (3.4). One can see that the fundamental solution is not a function in form of $U(t)V(x)$, thus the method of separation of variables won't work here. But it also shows that there are other types of solutions of diffusion equation which cannot be obtained by separation of variables.

When deriving the diffusion equation in Chapter 1 via random walk, an important assumption we make is that $(\Delta x)^2/(2\Delta t) \to D$ when the step sizes Δx and Δx approach zero. So we can guess that that the scaling x^2/t can be a factor in the solution of diffusion equation. Indeed, if $u(t, x)$ is a solution to the diffusion equation, so is $u(a^2t, ax)$, a dilation of $u(t,x)$. Thus we try a test function $u(t,x) = v(x^2/t)$, and hope it may solve the diffusion equation. Then

$$u_t = v'\left(\frac{x^2}{t}\right) \cdot \left(-\frac{x^2}{t^2}\right), \text{ and } u_{xx} = v''\left(\frac{x^2}{t}\right) \cdot \left(\frac{4x^2}{t^2}\right) + v'\left(\frac{x^2}{t}\right) \cdot \left(\frac{2}{t}\right). \tag{3.12}$$

Substituting (3.12) into (3.11), and after some algebra, we have

$$v''(y) + \frac{2D + y}{4Dy} v'(y) = 0. \tag{3.13}$$

By integrating, we find that

$$v'(y) = c_1 y^{-1/2} e^{-y/(4D)}, \tag{3.14}$$

thus the general solution of (3.13) is

$$v(y) = c_1 \int_0^y z^{-1/2} e^{-z/(4D)} dz + c_2, \tag{3.15}$$

for constant $c_1, c_2 \in \mathbf{R}$. So we now have another family of solutions of diffusion equation:

$$u(t,x) = v(x^2/t) = c_1 \int_0^{x^2/t} z^{-1/2} e^{-z/(4D)} dz + c_2. \tag{3.16}$$

However this solution is still not the fundamental solution. We notice that if $u(t,x)$ is a solution of the diffusion, so is its partial derivatives as the equation is linear. We differentiate (3.16) with respect to x, and we obtain

$$v(t,x) = \frac{2c_1}{\sqrt{t}} e^{-\frac{x^2}{4Dt}}. \tag{3.17}$$

Finally we choose $c_1 = M/(4\sqrt{\pi}D)$ so that $\int_{\mathbf{R}} v(t,x)dx = M$, and we obtain the formula of the fundamental solution.

There are at least another three ways of deriving the fundamental solutions. One is to use a test function $u(t,x) = t^{-\alpha} v(t^{-\beta} x)$ with unspecified α and β; the second is to use Fourier transform, an integral transform; and the third is to use random walk approach in Chapter 1 and the central limit theorem in probability theory.

The solutions given in (3.16) are also very useful in some cases. By making a change of variables $z = 4Dw^2$, and assuming that $u(t,x) \to 0$ as $x \to \pm\infty$, we can rewrite the solutions in (3.16) as

$$u(t,x) = c_3 \left(1 - \frac{2}{\pi} \int_0^{x/\sqrt{4Dt}} e^{-w^2} dw \right) = c_3[1 - erf(x/\sqrt{4Dt})], \tag{3.18}$$

where $erf(y) = \dfrac{2}{\pi} \displaystyle\int_0^y e^{-w^2} dw$ is called the error function, which is widely used in statistics and applied mathematics. The solution in (3.18) satisfies

$$\begin{cases} \dfrac{\partial u}{\partial t} = D \dfrac{\partial^2 u}{\partial x^2}, \quad t > 0, \ x \in (0, \infty), \\ u(t, 0) = c_3, \ \lim\limits_{x \to \infty} u(t,x) = 0, \\ u(0,x) = 0, \quad x \in (0, \infty). \end{cases} \tag{3.19}$$

The equation (3.19) is a diffusion on a half line with continuous source instead of a point source. Here $x = 0$ is the source, where the concentration or density of u is kept as a constant c_3.

3.3 Chemical Pollution: revisit

In Section 2.5, we consider a chemical mixing-diffusion problem in a tube with finite length L, and a continuous source flux at one end. Here we apply some results in two previous sections to consider some more chemical questions.

Example 3.1. Prof. Shi's house is on a long boat canal. One day a neighbor 200 meters away from his house has a small fuel spill (1 kg). Suppose that the current in the canal is negligible, and the fuel is only transported through diffusion. Assume that fuel mixes rapidly across the width and stays on the surface of the canal, and assume the canal is 10 meter wide and the diffusion constant $D = 0.04m^2/s$.

1. How long does it take for the spilled fuel to reach Prof. Shi's house?
2. What is the concentration of the fuel at Prof. Shi's house at that time?
3. When does the fuel achieve the maximum concentration at Prof. Shi's house? What is the maximum concentration?

Let $f(t,x)$ be the concentration of the fuel. Then f satisfies the diffusion equation, and the fuel spill can be thought as a point source release. The long canal can be assumed to be infinitely long. Thus f satisfies

$$f_t = Df_{xx}, \ \ t > 0, \ x \in \mathbf{R}, \ \ u(0,x) = \delta_0(x). \tag{3.20}$$

The solution is readily given by the fundamental solution:

$$\Phi(t,x) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}. \tag{3.21}$$

However to give a meaningful answer to the question, we have to define what means the fuel "reaches" the house. In fact, for any $t > 0$, $\Phi(t,x) > 0$ for all $x \in \mathbf{R}$, so the fuel technically reaches the house immediately, though only very tiny portion at a point far away from the source. One way to define a minimum traceable level k, and to calculate the time when the concentration at the given point reaches that level. But a conventional way is to define the edge of the diffusing patch of the fuel. We notice that $\Phi(t,x)$ is a normal distribution function with variance $\sigma = \sqrt{2Dt}$. The patch is usually defined as the spatial domain within $\pm 2\sigma$ of the center. Thus the edge is at $x = 2\sqrt{2Dt}$, and the time that the edge reaches the house can be calculated by

$$2\sqrt{2Dt} = 200, \tag{3.22}$$

and $t = 10^6/8 = 125000s \approx 32$ hours. The concentration at that time is

$$\Phi(125000, 2\sqrt{2Dt}) = \frac{1}{10\sqrt{4\pi \cdot 0.04 \cdot 125000}} e^{-2} = 5.4 \times 10^{-5}(kg/m^2) = 0.054(g/m^2). \tag{3.23}$$

The time 32 hours is when the edge of the patch reaches the house, but not the time the maximum concentration is at the house. To calculate the maximum concentration, we can use Maple to differentiate the solution with respect to t, and solve the time of maximum concentration: $t =$

$500000s \approx 128$ hours. The maximum concentration is about $0.121(g/m^2)$. We also notice that although when $t \to \infty$, the solution tends to zero, but the rate of approaching zero is extremely slow. For instant, the time of $f(t, 200)$ back to $0.054(g/m^2)$ is $6302115s \approx 1750$ hours! Thus even the peak has passed after 5 days of the spill, there is still a noticeable trace of the fuel in the period of 73 days. Now you can imagine how bad it would be if a whole fuel tank spills on a highway, or a oil vessel spills in the ocean.

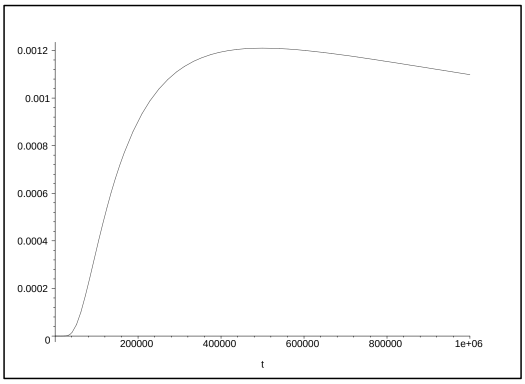

Figure 3.1: The graph of $f(t, 200)$

3.4 Diffusive Malthus equation and biological dispersion

The simplest population growth model is due to Thomas Malthus:

$$\frac{dP}{dt} = aP, \tag{3.24}$$

where a is the constant growth rate per capita. Thus if we take the spatial distribution into the consideration, it is natural to consider the following diffusive Malthus equation (in two-dimensional space):

$$\frac{\partial P}{\partial t} = D(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}) + aP, \tag{3.25}$$

where a is the constant growth rate per capita per unit area. In this section, we consider the equation in the whole space with a point source:

$$\begin{cases} \frac{\partial P}{\partial t} = D(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}) + aP, \quad t > 0, \ (x, y) \in \mathbf{R}^2, \\ u(0, x, y) = \delta_{(0,0)}(x, y). \end{cases} \tag{3.26}$$

The solution of (3.26) can easily deduced from the solution of (3.8) (the 2-D fundamental solution). In fact, let $\Phi(t, x, y)$ be the solution of (3.8), then it is easy to verify that $e^{at}\Phi(t, x, y)$ is the solution of (3.26). Thus the solution of (3.26) can be expressed as

$$P(t, x, y) = \frac{M}{4\pi Dt} e^{at - \frac{x^2+y^2}{4Dt}}, \tag{3.27}$$

and consequently

$$\int_{\mathbf{R}^2} P(t,x,y)dxdy = Me^{at}. \tag{3.28}$$

This equation can be used to model the population explosion of modern metropolitan area in twentieth century. Suppose that A is an isolated metropolitan area which is far away from any other metropolitan areas, then we can assume that A and the surrounding areas is an unlimited area—$\mathbf{R}^2$. Here we also assume that the migration of the population can be in any direction—there is no water bodies in the area, no unlivable mountain, desert, etc. At the time $t = 0$ (the beginning of the city), a group of M immigrants came to this place, and began the formation of the city by diffusion and growth. After some time, when a town has been formed near the center of the first settlement $(0,0)$, we can define the *metro area* and the *rural area* in the following way: Let B_R be the ball with radius R in $\mathbf{R}^2$; we define $R(t)$ as the number such that outside of $B_{R(t)}$, the total population is always M, *i.e.*

$$\int_{\mathbf{R}^2 - B_{R(t)}} P(t,x,y)dxdy = M; \tag{3.29}$$

we define $B_{R(t)}$ as the metro area and $\mathbf{R}^2 - B_{R(t)}$ as the rural area. We make such a definition in assuming that the population in all rural area keeps a constant despite the growth, but the population in metro area has an exponential growth, and the area of the metro also grows over years. With the calculation which leads to (3.28), we have

$$\int_{\mathbf{R}^2 - B_{R(t)}} P(t,x,y)dxdy = Me^{at}e^{\frac{R^2(t)}{4Dt}},$$

thus $R(t)$ and the area of $B_{R(t)}$ the form

$$R(t) = \sqrt{4aD}t, \quad A(t) = 4\pi aDt^2. \tag{3.30}$$

The boundary of $B_{R(t)}$ can be regarded as the front of population wave, and that is where new neighborhoods and new shopping centers are being built. The speed of the expanding of the metro area is

$$R'(t) = \sqrt{4aD}. \tag{3.31}$$

The analysis of the formation of the modern metropolitan has been carried out by Skellam [**?**] in early 1950's for the spreading of muskrats (*Ondatra zibethica*) in central Europe. Today the muskrat is quite common in Europe and Asia. Its range extends from Sweden and France in the west to the major river systems of Siberia in the east. But muskrats are a recent addition from the New World. According to a study published in 1930 by Ulbrich, referenced by Skellam [**?**], in 1905 several muskrats found their way to freedom in a wilderness near the Moldau River about 50 kilometers southwest of Prague. Radial dispersion and exponential growth followed. Although the muskrat has many natural predators, moat notably the mink, in this instance there was evidently no natural-imposed carrying capacity during the ensuing years. the muskrat population grew rapidly and dispersed widely.

Over the years, Ulbrich kept records of the spread of the muskrat. His map of the locations of the dispersion front is shown ?. Cumulative areas defined by the level curves (contours) of the map are listed in the following table.

Year	1905	1909	1911	1915	1920	1927
Area (km^2)	0	5400	14000	37700	79300	201600

We use a Maple program to fit the above data into a function $A(t) = kt^2$:

```
>restart;with(stats):with(plots):Times:=[0,4,6,10,15,22];
>Area:=[0,5400,14000,37700,79300,201600];
>Data:=[[0,0],[4,5400],[6,14000],[10,37700],[15,79300],[22,201600]];
>eq_fit:=fit[leastsquare[[x,y],y=k*x^2,{k}]]([Times,Area]);
```

$$eq_fit := y = \frac{119777300}{296433}x^2$$

Thus $aD = \frac{119777300}{296433} \cdot \frac{1}{4\pi} = 32.15422940$, and the speed of the expansion is $R'(t) = \sqrt{4aD} = 11.34093989(km/year)$. The equation (3.25) is the simplest model for biological invasion, and it gives good estimate of the speed of invasion wave in the initial stage of the dispersion. However it is also unrealistic in long term because of exponential growth. In the next Chapter we will discuss more realistic invasion models.

Chapter 3 Exercises

1. Consider the diffusion equation $u_t = Du_{xx}$. Assume that $u(t,x) = v(x/\sqrt{t})$, derive an ordinary differential equation satisfied by v, and show that the solution $u(t,x)$ is given by (3.17).

2. Consider the convective-diffusion equation:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} - V\frac{\partial u}{\partial x}, \quad t > 0, x \in \mathbf{R}, u(0,x) = M\delta_0(x). \tag{3.32}$$

   Use a change of variable $z = x - Vt$, $s = t$ to show that $v(s,z) = u(t,x)$ satisfies $v_s = Dv_{zz}$, and derive the solution formula of (3.32):

$$u(t,x) = \frac{M}{\sqrt{4\pi Dt}} exp\left(-\frac{(x-Vt)^2}{4Dt}\right). \tag{3.33}$$

3. In the fuel spill example in Section 3.3, if we assume now the canal current is $V = 0.01(m/s)$ from the spill point toward Prof. Shi's house, then $f(t,x)$ satisfies (3.32). Use the results in problem 2 to recalculate the questions in Example 3.1. In this case, the center of the fuel patch is moving with a velocity V.

4. In the fuel spill example in Section 3.3, we assume that the current is from Prof. Shi's house toward the spill point. Then it is possible that the fuel never reaches Prof. Shi's house if the current is fast enough. (Here "reach"means the house is inside the $\pm 2\sigma$ of the center of the patch.) Determine for which V that will happen.

5. (Spread of gypsy moths, [?]) Gypsy moths (*Lymantria dispar*) were brought to Massachusetts from Europe around 1870 in connection with silkworm development research. Needless to say, some of the moths escaped from breeding cages but somehow large growths and widespread dispersal were kept under control for a number of years. However, around 1900 there was a drastic increase of gypsy moth population in the Boston area which quickly spread to adjacent regions. By 1925 or so, when dispersal was finally halted, gypsy moths covered all of New England and parts of New York state and Canada. There was severe damage to forests throughout the region. The following are the cumulative areas corresponding to the dispersion fronts. According to the studies by Elton, there was no significant expansion of the front after 1925.

Year	1900	1905	1910	1915	1920	1925
Area (km^2)	1290	9080	26960	58840	79770	113320

Use the data above to estimate the value of aD and the year when the area was zero in this example. (Hint: modify the Maple program above, use function $a(x-b)^2$ instead of ax^2, and notice that Boston is a coastal city, so the spread areas are semicircular instead of circular.)