Sums, series, sequences

(From Calculus II)

$$\sum_{k=1}^{n} k = 1+2+\cdots+n = \frac{n(n+1)}{2}, \sum_{k=1}^{n} k^2 = 1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

Geometric series:

$$a_n = a_1 \cdot r^{n-1},\ S_n = \sum_{i=1}^{n} a_i = \frac{a_1(1-r^n)}{1-r},\ S = \sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} S_n = \frac{a_1}{1-r}$$

p-series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$: convergent when $p > 1$, divergent when $p \le 1$

(Taylor series) $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$, (Maclaurin series) $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}x^n$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1+x+x^2+x^3+\cdots+x^n+\cdots, \quad e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots,$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots,$$

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{1}{n}x^n = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Methods of finding sums: telescope series, differentiate and integrate the power series

Examples:

1. (1) Find a formula for $\sum_{k=1}^{n} k^3$; (2) Show that $\sum_{n=1}^{99} \frac{1}{\sqrt{n}+\sqrt{n+1}} = 9$.
2. Find the sum $\cos(\theta)+\cos(2\theta)+\cos(3\theta)+\cdots+\cos(n\theta)$. (Hint: use Euler's formula: $e^{ix} = \cos(x) + i\sin(x)$)
3. (UIUC 2003) Evaluate $\frac{1}{1\cdot 2\cdot 3} + \frac{1}{3\cdot 4\cdot 5} + \frac{1}{5\cdot 6\cdot 7} + \cdots$.
4. (1) $\sum_{n=0}^{\infty} \frac{(n+1)^2}{3^n}$; (2) $\sum_{n=0}^{\infty} \frac{(n+1)^2}{n!}$; (3) Evaluate $\sum_{r=1}^{\infty} \left(\sum_{s=1}^{r} s^2 \right)^{-1}$
5. (Putnam 1984 A-2) Express $\sum_{k=1}^{\infty} \frac{6^k}{(3^{k+1}-2^{k+1})(3^k-2^k)}$ as a rational number.
6. (Putnam 1999 A-4) Sum the series $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 n}{3^m(n3^m + m3^n)}$