Problem Set 3

Discussion Problems Discussion: Sept. 25

1. (UIUC 2003) Let $N = 9 + 99 + 999 + \cdots + \overbrace{99\cdots9}^{99}$. Determine the sum of digits of N. `(Sean)`
2. (UIUC 1998) Evaluate $\sum_{k=1}^{n} \frac{k}{2^{k-1}}$. `(David)`
3. (UIUC 2004) Let F_n denote the Fibonacci sequence, defined by $F_1 = 1$, $F_2 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$. Evaluate $\sum_{n=1}^{\infty} \frac{F_n}{3^n}$. `(Drew)`
4. (VT 2003) Evaluate $\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)} = \frac{x}{1\cdot 2} + \frac{x^2}{2\cdot 3} + \frac{x^3}{3\cdot 4} + \cdots$. `(Carolyn)`
5. (VT 1994-7) Define $f(1) = 1$ and $f(n+1) = 2\sqrt{[f(n)]^2 + n}$ for $n > 1$. If $N \geq 1$ is an integer, find $\sum_{n=1}^{N} [f(n)]^2$. `(Alexander)`
6. (Putnam 1977 B-1) Evaluate the infinite product: $\prod_{n=1}^{\infty} \frac{n^3 - 1}{n^3 + 1}$. `(Kassie)`
7. (Putnam 1978 B-2) Express $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m^2 n + mn^2 + 2mn}$ as a rational number. `(Katelyn)`
8. (Putnam 1977 A-4) For $0 < x < 1$, express $\sum_{n=0}^{\infty} \frac{x^{2^n}}{1 - x^{2^{n+1}}}$ as a rational function of x.

   (UIUC 2000) Evaluate $\frac{1}{2^1 - 2^{-1}} + \frac{1}{2^2 - 2^{-2}} + \frac{1}{2^4 - 2^{-4}} + \frac{1}{2^8 - 2^{-8}} + \cdots$ `(Katie)`

More Problems:

1. (VT 1992) Find $\lim_{n\to\infty} \frac{2\log 2 + 3\log 3 + \cdots + n\log n}{n^2 \log n}$
2. (Putnam 1997 A-3) Evaluate
$$\int_0^{\infty} \left(x - \frac{x^3}{2} + \frac{x^5}{2\cdot 4} - \frac{x^7}{2\cdot 4\cdot 6} + \cdots\right)\left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} + \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \cdots\right) dx$$
3. (Putnam 1996 B-2) Show that for every positive integer n,
$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1\cdot 3\cdot 5\cdots(2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}$$
4. (Putnam 1986 A-3) Evaluate $\sum_{n=0}^{\infty} \cot^{-1}(n^2 + n + 1)$
5. (Putnam 2001 B-3) For any positive integer n, let $\langle n \rangle$ denote the closest integer to $\sqrt{n}$. Evaluate
$$\sum_{n=1}^{\infty} \frac{2^{\langle n\rangle} + 2^{-\langle n\rangle}}{2^n}.$$
6. (Putnam 2004 B-5) Evaluate
$$\lim_{x\to 1^-} \prod_{n=0}^{\infty} \left(\frac{1 + x^{n+1}}{1 + x^n}\right)^{x^n}.$$