Leslie matrix model

$$L = \begin{pmatrix} s_1 m_1 & s_1 m_2 & \cdots & s_1 m_{k-1} & s_1 m_k \\ s_2 & 0 & \cdots & 0 & 0 \\ 0 & s_3 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & s_k & 0 \end{pmatrix}$$

$z(n) = (z_1(n), z_2(n), \cdots, z_k(n))^T$ is the population vector with $z_i(n)$ being the population of the i-th age group at time n. $z_1(n)$ is the population of the youngest group, and $z_k(n)$ is that of the oldest group. Let s_i be the survival rate from age $i-1$ to i, $0 < s_i < 1$ for $1 \leq i \leq k$;

let m_i be the average reproduction rate per capita for age i.

$m_i > 0$ only for $\alpha \leq i \leq \beta$ (fertile period)

Consider solving the equation: take $n = 4$ for simplicity

$b(n)$ =total birth rate= $\sum_{i=1}^{k} m_i u_i(n-1)$

$$\begin{pmatrix} u_1(n) \\ u_2(n) \\ u_3(n) \\ u_4(n) \end{pmatrix} = \begin{pmatrix} s_1 b(n) \\ s_2 s_1 b(n-1) \\ s_3 s_2 s_1 b(n-2) \\ s_4 s_3 s_2 s_1 b(n-3) \end{pmatrix} = \begin{pmatrix} l_1 b(n) \\ l_2 b(n-1) \\ l_3 b(n-2) \\ l_4 b(n-3) \end{pmatrix}, \text{ for } n \geq 5$$

Define $l_i = s_1 s_{i-1} \cdots s_3 s_2 s_1$ to be the survival function, which is the fraction of population surviving form birth to age i.

Renewal equation:

$$b(n+1) = \sum_{i=1}^{k} l_i m_i b(n+1-i) \text{ or } b(n) = \sum_{i=1}^{k} l_i m_i b(n-i)$$

$f_i = l_i m_i$ is the net maternity function, the number of offsprings produced by each female of age i

$b(n) = \sum_{i=1}^{k} f_i b(n-i)$, k-th order linear difference equation
characteristic equation for eigenvalue:
$\lambda^k = f_1 \lambda^{k-1} + \cdots + f_{k-1}\lambda + f_k$
eigenvalues: $\lambda_1, \lambda_2, \cdots, \lambda_k$
$b(n) = c_1 \lambda_1^n + c_2 \lambda_2^n + \cdots + c_k \lambda_k^n$

$u_i(n)$ can be solved by $b(n)$ since $u_i(n) = l_i b(n-i)$

Life Table http://en.wikipedia.org/wiki/Life_table

In actuarial science, a life table (also called a mortality table or actuarial table) is a table which shows, for a person at each age, what the probability is that they die before their next birthday. From this starting point, a number of statistics can be derived and thus also included in the table:

1. the probability of surviving any particular year of age
2. remaining life expectancy for people at different ages
3. the proportion of the original birth cohort still alive.

Life tables are usually constructed separately for men and for women because of their substantially different mortality rates. Other characteristics can also be used to distinguish different risks, such as smoking-status, occupation, socio-economic class, and others.

Life tables are also used in biology.

Example 1 Semelparous reproduction

Semelparous organisms reproduce only once in their lifetime, such as annual plants. Often, they die shortly after reproduction.

Life table:

i	l_i	m_i	s_i	f_i
1	1	0	1	0
2	1/2	0	1/2	0
3	1/6	6	1/3	1

Leslie matrix: $\begin{pmatrix} 0 & 0 & 6 \\ 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \end{pmatrix}$

Renewal equation: $\lambda^3 = 1$,

Eigenvalues: $\lambda_1 = 1$, $\lambda_{2,3} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}$

Harro Bernardelli, Population waves, 1941
(Journal of the Burma Research Society)
oscillation of Burmese population from 1901 to 1931

Example 2 US population in 1966

(N Keyfitz, W Flieger, 1971, Population: Facts and Methods of Demography)

i	1	2	3	4	5
$s_1 m_i$	0	0.00102	0.08515	0.30574	0.40002
s_i	-	0.99670	0.99837	0.99870	0.99672

i	6	7	8	9	10
$s_1 m_i$	0.28061	0.15260	0.06420	0.01483	0.00089
s_i	0.99607	0.99472	0.99240	0.98867	0.98274

5-year age classes

Population growth from life table:

$\lambda_1 = 1.0498$ (growth rate per capita 4.98%

eigenvector: (stable population distribution)

$(0.1252, 0.1189, 0.1131, 0.1075, 0.1020, 0.0968,$

$0.0917, 0.0867, 0.0817, 0.0765)$

Work on age structure

Leonhard Euler (1760) Alfred Lotka (1880-1949)

`http://en.wikipedia.org/wiki/Leonhard_Euler`

`http://en.wikipedia.org/wiki/Alfred_J._Lotka`