Paradiso, Purgatory, and Inferno model:

$$\begin{pmatrix} x_1(n+1) \\ x_2(n+1) \\ x_3(n+1) \end{pmatrix} = \begin{pmatrix} 1.5(1-d) & 0.5d & 0.1d \\ 0.75d & 1-d & 0.1d \\ 0.75d & 0.5d & 0.2(1-d) \end{pmatrix} \begin{pmatrix} x_1(n) \\ x_2(n) \\ x_3(n) \end{pmatrix},$$
$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} 100 \\ 200 \\ 300 \end{pmatrix}.$$

Shorter form: $\vec{x}(n+1) = A \cdot \vec{x}(n)$, $\vec{x}(0) = (100; 200; 300)$.

The solution:
$x(n) = Ax(n-1) = A^2x(n-2) = A^3x(n-3) = \cdots = A^n x(0)$
(here A^n is the power of matrix A, which is not easy to calculate, but easy for Matlab)

Observation from Matlab simulation:

$d = 0.2$

1. All population grow.
2. The total population grows exponentially.
(log(P) has a linear growth)
3. The fraction of each population in the total population tends to a constant

$d = 0.8$

1. All population decay.
2. The total population decays exponentially.
3. The fraction of each population in the total population tends to a constant

A stage structured model:

$$x(n+1) = \begin{pmatrix} 0 & 1.1 \\ 0.55 & 0.55 \end{pmatrix} x(n),\ x(0) = (200; 0).$$

Observation: exponential growth, the numbers of juveniles and adults are almost same after a few time units.

For $x(0) = \begin{pmatrix} 100 \\ 100 \end{pmatrix}$, then $x(n) = A^n \begin{pmatrix} 100 \\ 100 \end{pmatrix} = 1.1^n \begin{pmatrix} 100 \\ 100 \end{pmatrix}$

$$A \begin{pmatrix} 100 \\ 100 \end{pmatrix} = 1.1 \begin{pmatrix} 100 \\ 100 \end{pmatrix}.$$

so the matrix power could be just the power of a scalar number. Linear algebra can explain all these. (Math 211)

A crash course of linear algebra

The numbers λ satisfying $Ax = \lambda x$ are called <u>eigenvalues</u>.
The corresponding $x (\neq 0)$ is called <u>eigenvector</u> associated with the eigenvalue.
An $n \times n$ matrix has exactly n eigenvalues (they could be same, but with different eigenvectors)
If v is an eigenvector, so is cv (c is a constant)

Example: 2×2 matrix
Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 2 & 1 \\ 4 & 5 \end{pmatrix}$

Eigenvalue and eigenvector of 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

solve $(a-\lambda)(d-\lambda) - bc = 0$ (characteristic equation) for eigenvalues λ_1, λ_2

Example: Fibonacci model: $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

Eigenvalues, eigenvectors for higher dimensional matrices: use Matlab to solve

Solution of $x(n+1) = Ax(n)$ (A is a $k \times k$ matrix):

$$x(n) = c_1\lambda_1^n v_1 + c_2\lambda_2^n v_2 + \cdots + c_k\lambda_k^n v_k = \sum_{i=1}^{k} c_i\lambda_i^n v_i$$

where λ_i are eigenvalues, v_i are eigenvectors, and c_i are constants.

The eigenvalues can be ordered so that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_k|$, and λ_1 is called dominant eigenvalue.

$x(n) \approx c_1\lambda_1^n v_1$ for large n

Stability

If $|\lambda_1| < 1$, then $x(n) \to 0$ as $n \to \infty$ (extinction)

If $|\lambda_1| > 1$, then $x(n) \to \infty$ as $n \to \infty$ and $x(n) \approx c_1 \lambda_1^n v_1$

$\dfrac{1}{\lambda_1^n} x(n) \approx c_1 v_1$

Let $w_1 = \dfrac{v_1}{sum(v_1)}$, then w_1 is the stable state distribution of the model

Perron-Frobenius Theorem: If A has non-negative entries and is power positive (there is a natural number n such that A^n has only positive entries), then the dominant eigenvalue of A is positive, and the associated eigenvector is also positive.