Fibonacci Fibonacci is the "greatest European mathematician of the middle ages", his full name was Leonardo of Pisa, or Leonardo Pisano in Italian since he was born in Pisa (Italy), the city with the famous Leaning Tower, about 1175 AD.He was one of the first people to introduce the Hindu-Arabic number system into Europe - the positional system we use today - based on ten digits with its decimal point and a symbol for zero: 1234567890 His book on how to do arithmetic in the decimal system, called Liber abbaci (meaning Book of the Abacus or Book of Calculating) completed in 1202 persuaded many European mathematicians of his day to use this "new" system.



Fibonacci's rabbit problem In Fibonacci's Liber Abaci (written in 1202), chapter 12, he introduces the following problem:

How Many Pairs of Rabbits Are Created by One Pair in One Year? A certain man had one pair of rabbits together in a certain enclosed place, and one wishes to know how many are created from the pair in one year when it is the nature of them in a single month to bear another pair, and in the second month those born to bear also.

1. Initially there is one pair of rabbits;

2. Each pair of rabbits can reproduce since they are two months old, (not not one month old).

Recursive relation: $y_{n+1} = y_n + y_{n-1}$, $y_0 = y_1 = 1$ (y_n is the number of pair of rabbits after n months) $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144 \cdots$

Guess a solution: exponential solution $y_n = CR^n$ then $R^2 = R + 1$, and $R = \frac{1 \pm \sqrt{5}}{2}$,

$$y_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \approx \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$$

Fibonacci sequence in nature (leaf patterns, pine cone spiral) http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fibnat.html

What difference does it make to the model if the census takes place just after the births instead of just before?

2nd order linear difference equation

 $y_{n+1} = \lambda y_n$ solution $y_n = y_0 \lambda^n$ (exponential) $y_{n+1} = ay_n + by_{n-1}$ solution $y_n = c_1 \lambda_1^n + c_2 \lambda_2^n$ (c_1 and c_2 are determined by the initial conditions y_0 and y_1

 λ_1 , λ_2 are solutions of $\lambda^2 = a\lambda + b$, and these two numbers are called eigenvalues of the problem, and $\lambda^2 = a\lambda + b$ is the characteristic equation.

Example: Solve $y_{n+1} = 5y_n + y_{n-1}$, $y_0 = 1$ and $y_1 = 2$.

 $y_{n+1} = ay_n + by_{n-1} + c$: try a solution in form of $y_n = c_1\lambda_1^n + c_2\lambda_2^n + e$ (e is the fixed point or equilibrium solution)

Age structured models

Fibonacci's rabbit model not only considers the total number of rabbits, but also the ages of rabbit. We can reformat the model in this way: let M_n be the number of adult pairs of rabbits (at least two months old), and let J_n be the number of juvenile pair (one month old), then

$$M_{n+1} = M_n + J_n, \ J_{n+1} = M_n, \text{ with } M_0 = 0 \text{ and } J_n = 1,$$

or in matrix notation:
$$\begin{pmatrix} J_{n+1} \\ M_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} J_n \\ M_n \end{pmatrix}, \text{ and } \begin{pmatrix} J_0 \\ M_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

This is a matrix model in discrete time.

Another example: Moving between Paradiso, Purgatory, and Inferno Consider a population that disperses between three habitats; habitat 1 that is of very high quality (paradiso), habitat 2 that is of intermediate quality (purgatory), and habitat 3 that is of very low quality (inferno). The populations are censused prior to the adults reproducing. In all habitats, all females produce two daughters and then die. The fraction of daughters surviving in patches 1, 2, and 3 are 0.75 (not quite paradise), 0.5, and 0.1, respectively. After surviving, a constant fraction of progeny ddisperse from their current habitat and go with equal likelihood to either of the two other habitats. Write down a difference equation model for the females in this population. Determine what happens after one generation if initially there were 100 females in each habitat and d = 0.2.