Dimensions:

Dimension is the unit of certain type of physical quantity.

Example: the dimension of variable t (time) is second or hour, the dimension of variable L (length) is meter or foot, and the dimension of parameter N (carrying capacity) is million people.

Dimensionless: A number is dimensionless if it is just a number, it is not a measurement of any type of physical quantity.

Example: Quotient of N (carrying capacity) and P (the population variable) is a dimensionless quantity.

Every variable or parameter in the differential equation has a dimension or is dimensionless.

Nondimensionalization:

Nondimensionalization is a process of changing variables by scaling so that the new variables are dimensionless, and it leads to a simpler form of the equation with fewer parameters.

Step 1: List all variables, parameters and their dimensions.

Step 2: Take each variable and create a new variable by dividing by the combination of parameters that has the same dimension in order to create a dimensionless variable. (Usually there is more than one way to do this.

Step 3: Calculate and simplify the new equation via change of variables.

Step 4: Introduce new parameters.

Example 1:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right)\left(\frac{P}{M} - 1\right).$$ (1)

Variable	Dimension	Parameter	Dimension
t	τ	k	
P	ρ	M	
		N	

Example 2:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right) - \frac{BP^2}{A^2 + P^2}, \quad P(0) = P_0. \qquad (2)$$

| Variable	Dimension	Parameter	Dimension
t	τ	k	$1/\tau$
P	ρ	N	ρ
		A	ρ
		B	ρ/τ
		P_0	ρ

Bifurcation: Suppose that the differential equation depends on a parameter. Then we say that a bifurcation occurs if there is a qualitative change in the behavior of solutions as the parameter changes.

Example 1: $\dfrac{dy}{dt} = ky(1 - y)$ (no bifurcation)

Example 2: $\dfrac{dy}{dt} = y^2 - \mu$ (saddle-node bifurcation, supercritical)

Example 3: $\dfrac{dy}{dt} = y^3 + \mu y$ (pitchfork bifurcation, subcritical)

Example 4: $\dfrac{dy}{dt} = y^2 - \mu y$ (transcritical bifurcation)
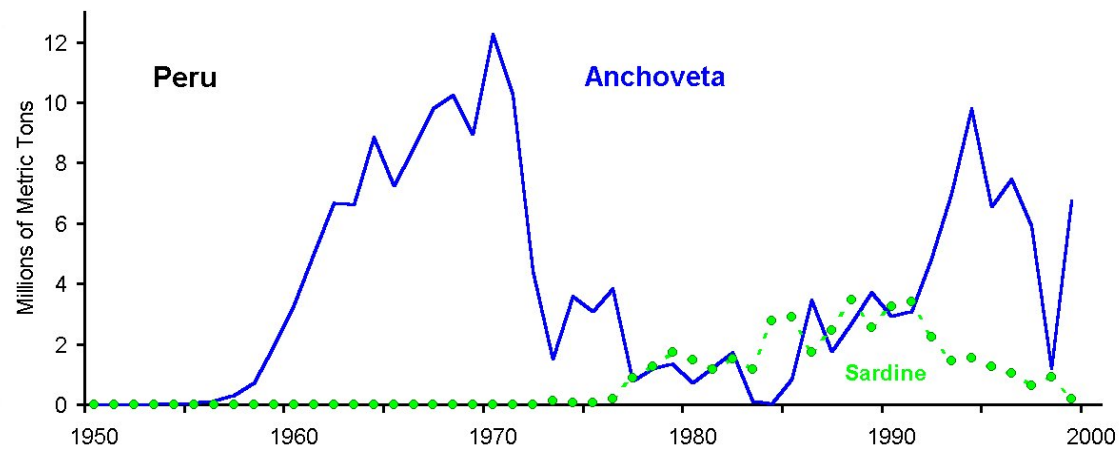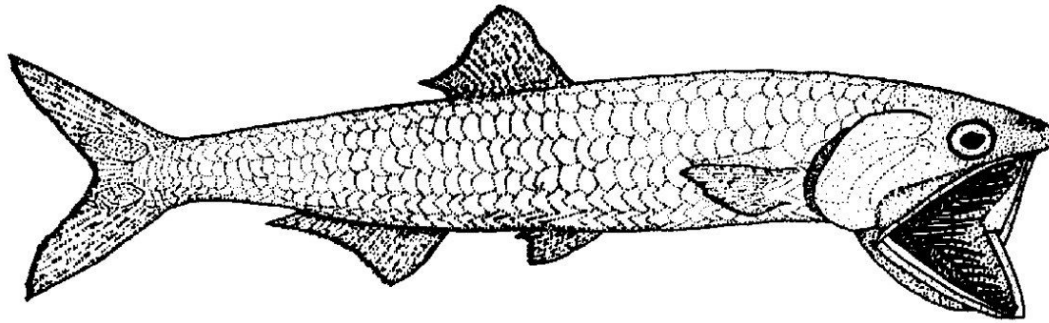
The Peruvian Anchovy Fishery

Peruvian Anchovy Fishery Before 1950, fish in Peru were harvested mainly for human consumption. The total annual catch was 86,000 tons. In 1953, the first fish meal plants were developed. Within 9 years, Peru became the number one fishing nation in the world by volume. This lead to a period of boom years in Peru. 1,700 purse seiners exploited a 7-month fishing season.

Fearing a crash, in 1970, a group of scientists in the Peruvian government issued a warning. They estimated that the sustainable yield was around 9.5 million tons, a number that was currently being surpassed. The government turned a deaf ear toward its own scientists. Due to the collapse of the Norwegian and Icelandic herring fisheries the previous year, Peru was more poised than ever to earn yet more hard currency. Therefore, in 1970, the government allowed a harvest of 12.4 million tons. The following year, 10.5 million tons were harvested. In 1972, the combination of an El Nino year and the prolonged overfishing led to a complete collapse of the fishery. It has not recovered.

Example 5: Constant yield harvesting

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right) - h$$

Mathematical Analysis:

1. Nondimensionalization: $u = \dfrac{P}{N}, s = kt,$

$$\frac{du}{dt} = u(1-u) - H, \quad H = \frac{h}{kN}$$

2. Bifurcation: a subcritical saddle-node bifurcation occurs at $H = 0.25$, or $h = 0.25kN$.

3. Qualitative analysis: when $0 < H < 0.25$, $H = 0.25$ and $H > 0.25$.

4. Analytic method: solve the equation? (you could...)

Example 5 (Cont.): Constant yield harvesting

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right) - h$$

Biological interpretation:

1. When $0 < H < 0.25$, there are two equilibrium points $P_1 > P_2 > 0$; for $P(0) > P_2$, $\lim_{t\to\infty} P(t) = P_1$ and for $0 < P(0) < P_2$, $P(t) < 0$ for $t > t_0$; P_1 is smaller than N, that means the carrying capacity decreases because of harvesting; the behavior of solutions with $P(0) > P_2$ is similar to that of logistic equation; if the initial population is less than P_2, then the population becomes extinct in finite time.

2. When $H > 0.25$, there is no equilibrium points, and for any initial population, the population becomes extinct in finite time.

3. $H = 0.25$ or $h = 0.25kN$ is called Maximum Sustainable Yield (MSY).

Example 6: Constant effort harvesting

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right) - hP$$

Mathematical Analysis:

1. Nondimensionalization: $u = \dfrac{P}{N}, s = kt,$

$$\frac{du}{dt} = u(1 - u) - Hu, \quad H = \frac{h}{kN}$$

2. Bifurcation: no bifurcation if we assume that $0 < H < 1$

3. A different question: for which H, we can get the maximum yield Hu?