Half life: $P' = -kP$, $P(0) = P_0$, the time T so that $P(T) = P_0/2$
Example Polonium-210 has a half-life of 140 days.
(1) If a sample has a mass of 50 mg, find the mass after t days.
(2) Find the mass after 100 days.

Example *Determining an infusion rate* An asthmatic patient is given a continuous infusion of theophylline to relax and open the air passages in his lungs. The desired steady-state level of theophylline in the patients blood stream is 15 mg/L. The average half-life of theophylline is about 4 hours, and the patient has 5.6 liters of blood. Find the infusion required to maintain the desired steady-state level.

Linear model: $\frac{dP}{dt} = aP + b$, $P(0) = P_0$
solution in a form of $P(t) = P_* + (P_0 - P_*)e^{at}$, where P_* is the equilibrium solution

Direction field: (slope field)

$$\frac{dP}{dt} = f(t, P)$$

If the function $P(t)$ is a solution of the equation and if its graph passes through the point (t_0, P_0) where $P_0 = y(t_0)$, then the differential equation says that the derivative dP/dt at $t = t_0$ is given by the number $f(t_0, P_0)$.

Direction field: at each point selected, we draw a minitangent line (we call it slope mark) whose slope is $f(t, P)$.

Matlab program: dfield `http://math.rice.edu/~dfield/`
Java Version (run online!): `http://math.rice.edu/~dfield/dfpp.html`

Autonomous Equation:

$$\frac{dP}{dt} = f(P)$$

The direction field of an autonomous equation does not depend on t, so the slope marks at (t_1, P) and (t_2, P) are same. All solutions are parallel in P direction in the sense that if $P(t)$ is a solution, then $P(t+c)$ is also a solution. So we only need to draw slope marks for a fixed t.

Phase line: Phase line is a simplified direction field for autonomous equation only. On a vertical line, we indicate the positive derivative by an arrow pointing up, and negative derivative by an arrow pointing down.

Equilibrium solutions: $\dfrac{dP}{dt} = f(P)$

If for some number a, $f(a) = 0$, then $P(t) = a$ is an equilibrium solution (constant solution).

How to draw the phase line:

(1) Draw the P-line. (usually vertical)
(2) Find and mark the equilibrium points.
(3) Find the intervals of P-values for which $f(P) > 0$, and draw arrows pointing up in these intervals.
(4) Find the intervals of P-values for which $f(P) < 0$, and draw arrows pointing down in these intervals.

Example 1. (1) Draw the phase line of $P' = P(1 - P)(2 - P)$;
(2) Determine the long time behavior of the solutions with $P(0) = 0.3$ and $P(0) = 1.4$.

Stability of an equilibrium point:

Suppose that $y = y_0$ is an equilibrium point of $y' = f(y)$.

y_0 is a **sink** if any solution with initial condition close to y_0 tends toward y_0 as t increase.

y_0 is a **source** if any solution with initial condition close to y_0 tends toward y_0 as t decrease.

y_0 is a **node** if it is neither a sink nor a source.

Linearization Theorem:

Suppose that $y = y_0$ is an equilibrium point of $y' = f(y)$.

- if $f'(y_0) < 0$, then y_0 is a sink (exponential stable);
- if $f'(y_0) > 0$, then y_0 is a source (exponential unstable);
- if $f'(y_0) = 0$, then y_0 can be any type, but in addition
  - if $f''(y_0) > 0$ or $f''(y_0) < 0$, then y_0 is a node.

Model 3: Allee effect:

Assumptions:

(1) If the population is too large, then the growth is negative.
(2a) If the population is too small, then the growth is negative.

Example: The fox squirrel (a small mammal native to the Rocky Mountains.) If the population is too small, fertile adults run the risk of not being able to find suitable mates, so the growth is negative.

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)\left(\frac{P}{M} - 1\right)$$

$r =$ growth rate per capita coefficient, $K =$ carrying capacity, $M =$ sparsity constant, and $0 < M < K$.

Basic growth models:

Logistic model: $P' = rP\left(1 - \frac{P}{K}\right)$,
growth rate per capita $\frac{P'}{P} = r\left(1 - \frac{P}{K}\right)$ (decreasing)
maximum growth rate at zero population (**compensatory**)

Allee effect: $\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)\left(\frac{P}{M} - 1\right)$
growth rate per capita $\frac{P'}{P} = r\left(1 - \frac{P}{K}\right)\left(\frac{P}{M} - 1\right)$ (increasing, then decreasing)
maximum growth rate at a positive population, (**depensatory**)
critical depensation: growth is negative for small population

Biological reasons for Allee effect:

shortage of mates in sexual reproduction
predator satuartion
cooperative bahavior
lack of pollination

Pollen limitation causes an Allee effect in a wind-pollinated invasive grass (Spartina alterniflora), H.G. David, et. al., Proceedings of National Academy of Sciences, September 21, 2004, vol. 101, no. 38, 13804-13807.