Theorem: The $\beta'(b,b)$ distribution converges to the normal distribution when $b \to \infty$.

Proof[1]**:** The beta-prime distribution with the shape parameters α and β, $X \sim \beta'(\alpha,\beta)$, has the probability density function

$$f_X(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1+x)^{-\alpha-\beta}, \quad x \in (0,\infty). \tag{1}$$

For the special case of $\alpha = \beta = b$, $X \sim \beta'(b,b)$, the probability density function becomes

$$f_X(x) = \frac{\Gamma(2b)}{\Gamma(b)\Gamma(b)} x^{b-1}(1+x)^{-2b}. \tag{2}$$

We introduce the transformed random variable Y and the according Jacobian

$$Y = \sqrt{\frac{b}{2}}X - 1, \quad X = 1 + \sqrt{\frac{2}{b}}Y, \quad \frac{\mathrm{d}X}{\mathrm{d}Y} = \sqrt{\frac{2}{b}}. \tag{3}$$

For $y \in \left(\sqrt{\frac{2}{b}}, \infty\right)$, Y has the probability density function

$$\begin{aligned}
f_Y(y) &= \frac{\Gamma(2b)}{\Gamma(b)\Gamma(b)}\sqrt{\frac{2}{b}}\left(1+\sqrt{\frac{2}{b}}y\right)^{b-1}\left(2+\sqrt{\frac{2}{b}}y\right)^{-2b} \\
&= \frac{\Gamma(2b)}{\Gamma(b)\Gamma(b)}\sqrt{\frac{2}{b}}\left(1+\sqrt{\frac{2}{b}}y\right)^{-1}\left(\frac{1+\sqrt{\frac{2}{b}}y+\left(\frac{1}{2b}y^2-\frac{1}{2b}y^2\right)}{4+4\sqrt{\frac{2}{b}}y+\frac{2}{b}y^2}\right)^{b} \\
&= \frac{\Gamma(2b)}{\Gamma(b)\Gamma(b)}4^{-b}\sqrt{\frac{2}{b}}\left(1+\sqrt{\frac{2}{b}}y\right)^{-1}\left(1+\frac{-\frac{1}{2}y^2}{b\left(1+\sqrt{\frac{2}{b}}y+\frac{1}{2b}y^2\right)}\right)^{b}.
\end{aligned} \tag{4}$$

We can use Stirling's formula $\Gamma(z) = \sqrt{\frac{2\pi}{z}}\left(\frac{z}{e}\right)^z\left(1+O\left(\frac{1}{z}\right)\right)$ to rewrite the probability density function as

$$\begin{aligned}
f_Y(y) &= \frac{\sqrt{\frac{2\pi}{2b}}\left(\frac{2b}{e}\right)^{2b}}{\frac{2\pi}{b}\left(\frac{b}{e}\right)^{2b}}4^{-b}\sqrt{\frac{2}{b}}\left(1+\sqrt{\frac{2}{b}}y\right)^{-1}\left(1+\frac{-\frac{1}{2}y^2}{b\left(1+\sqrt{\frac{2}{b}}y+\frac{1}{2b}y^2\right)}\right)^{b}\left(1+O\left(\frac{1}{b}\right)\right) \\
&= \frac{1}{\sqrt{2\pi}}\left(1+\sqrt{\frac{2}{b}}y\right)^{-1}\left(1+\frac{-\frac{1}{2}y^2}{b\left(1+\sqrt{\frac{2}{b}}y+\frac{1}{2b}y^2\right)}\right)^{b}\left(1+O\left(\frac{1}{b}\right)\right).
\end{aligned} \tag{5}$$

With $z = -\frac{1}{2}y^2$ and $c = -\frac{1}{2}$, the limit of the central part of Equation (5) can be rewritten as

$$\lim_{b\to\infty}\left(1+\frac{z}{b\left(1+O\left(b^c\right)\right)}\right)^b = \exp\left(\lim_{b\to\infty}\frac{1}{b^{-1}}\ln\left(1+\frac{z}{b\left(1+O\left(b^c\right)\right)}\right)\right) = e^z, \tag{6}$$

[1] Following the ideas for the beta distribution from: R. Ryder, "Theorem: The beta(b,b) distribution converges to the normal distribution when $b \to \infty$," 2012. [Online]. Available: `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/BetaNormal.pdf`.

Proof for Equation (6): With $c < 0$, the inner limit is of indetermined form. The limit can be found by applying L'Hôpital's rule:

$$\lim_{b\to\infty} \frac{1}{b^{-1}} \ln\left(1 + \frac{z}{b\left(1+O\left(b^c\right)\right)}\right) = \lim_{b\to\infty} \frac{1}{\left(\frac{\mathrm{d}}{\mathrm{d}b} b^{-1}\right)} \left(\frac{\mathrm{d}}{\mathrm{d}b} \ln\left(1 + \frac{z}{b\left(1+O\left(b^c\right)\right)}\right)\right)$$
$$= \lim_{b\to\infty} -\frac{1}{b^{-2}} \frac{b\left(1+O\left(b^c\right)\right)}{b\left(1+O\left(b^c\right)\right) + \frac{b}{b} z} \frac{-z\left(1+O\left(b^c\right)\right)}{b^2\left(1+O\left(b^c\right)\right)} = \lim_{b\to\infty} \frac{z\left(1+O\left(b^c\right)\right)}{\left(1+O\left(b^c\right)\right)} = z. \tag{7}$$

The limit of the product of functions is the product of the limits of the functions unless we have an indeterminate form. For the analysis with $b \to \infty$ and fixed y, the limit of $\lim_{b\to\infty} \left(1 + \sqrt{\frac{2}{b}} y\right)^{-1} = 1$ and $\lim_{b\to\infty} \left(1 + O\left(\frac{1}{b}\right)\right) = 1$ can simply be dropped in Equation (5). Based on this and Equation (6), the probability density function of Y converges point wise to the probability density function of a standard normal random variable

$$\lim_{b\to\infty} f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}, \quad y \in \left(\sqrt{\frac{2}{b}}, \infty\right). \tag{8}$$

By Scheffé's theorem, Y converges in distribution to the standard normal distribution. By undoing the substitution, we can see that X converges in distribution to the normal distribution with mean 1 and variance $\frac{2}{b}$.