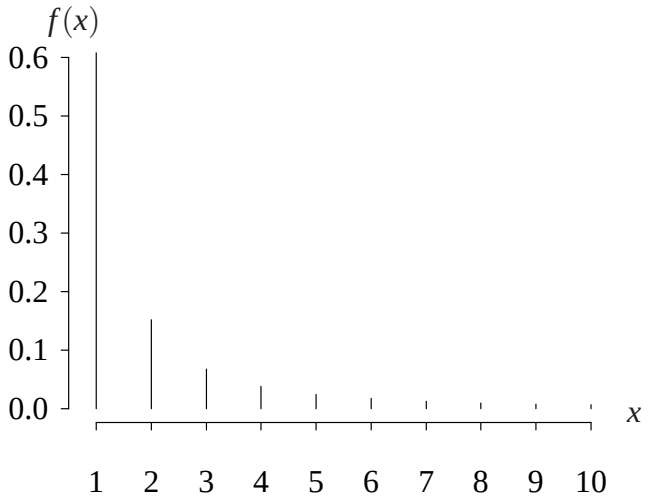
**Zeta distribution** (from http://www.math.wm.edu/~leemis/chart/UDR/UDR.html)
The shorthand $X \sim \text{Zeta}(\alpha)$ is used to indicate that the random variable $X$ has the Zeta distribution with parameter $\alpha > 1$. A Zeta random variable $X$ with parameter $\alpha$ has probability mass function

$$f(x) = \frac{1}{x^{\alpha} \sum_{i=1}^{\infty} (1/i)^{\alpha}} \qquad x = 0, 1, 2, \ldots$$

for any $\alpha > 1$. The probability mass function for $\alpha = 2$ is illustrated below.




The probability mass function can also be expressed as

$$f(x) = \frac{1}{x^{\alpha} \zeta(\alpha)} \qquad x = 1, 2, \ldots,$$

where $\zeta(\cdot)$ is the Riemann zeta function defined as

$$\zeta(\alpha) = \sum_{i=1}^{\infty} (1/i)^{\alpha}.$$

The cumulative distribution function on the support of $X$ is

$$F(x) = P(X \leq x) = \frac{\sum_{i=1}^{x} (1/i)^{\alpha}}{\zeta(\alpha)} \qquad x = 1, 2, \ldots .$$

The survivor function on the support of $X$ is

$$S(x) = P(X \geq x) = \frac{\sum_{i=x}^{\infty} (1/i)^{\alpha}}{\zeta(\alpha)} \qquad x = 1, 2, \ldots .$$

The hazard function on the support of $X$ is

$$h(x) = \frac{f(x)}{S(x)} = \frac{1}{\sum_{i=x+1}^{\infty}(1/i)^{\alpha}} \qquad x = 1, 2, \ldots .$$

The cumulative hazard function on the support of $X$ is

$$H(x) = -\ln S(x) = \ln(\zeta(\alpha)) - \ln\left(\sum_{i=x}^{\infty}(1/i)^{\alpha}\right) \qquad x = 1, 2, \ldots .$$

The inverse distribution function of $X$ is mathematically intractable.
The moment generating function of $X$ is

$$M(t) = E\left[e^{tX}\right] = \frac{1}{\zeta(\alpha)}\sum_{x=1}^{\infty}\frac{e^{tx}}{x^{\alpha}} \qquad x = 1, 2, \ldots .$$

The characteristic function of $X$ is

$$\phi(t) = E\left[e^{itX}\right] = \frac{1}{\zeta(\alpha)}\sum_{x=1}^{\infty}\frac{e^{itx}}{x^{\alpha}} \qquad x = 1, 2, \ldots .$$

The population mean and variance of $X$ are

$$E[X] = \frac{\zeta(\alpha-1)}{\zeta(\alpha)} \qquad \alpha > 2,$$

$$V[X] = \frac{\zeta(\alpha)\zeta(\alpha-2) - \zeta(\alpha-1)^2}{\zeta(\alpha)^2} \qquad \alpha > 3.$$

**APPL verification:** The APPL statements

```
assume(alpha > 1);
X := [[x -> 1 / (x ^ alpha * sum((1 / i ^ alpha), i = 1 .. infinity))],
      [0 .. infinity], ["Discrete", "PDF"]];
CDF(X);
SF(X);
HF(X);
CHF(X);
IDF(X);
MGF(X);
Mean(X);
Variance(X);
Skewness(X);
Kurtosis(X);
```

verify the cumulative distribution, survivor function, hazard function, cumulative hazard function, inverse, moment generating function, population mean, variance, skewness, kurtosis.