Theorem If $X \sim U(0,1)$, then $Y = -\alpha \ln X$ is exponentially distributed with mean α.

Proof Let the random variable X have the standard uniform distribution with probability density function

$$f_X(x) = 1 \qquad\qquad 0 < x < 1.$$

The transformation $Y = g(X) = -\alpha \ln X$ is a 1–1 transformation from $\mathcal{X} = \{x \mid 0 < x < 1\}$ to $\mathcal{Y} = \{y \mid y > 0\}$ with inverse $X = g^{-1}(Y) = e^{-Y/\alpha}$ and Jacobian

$$\frac{dX}{dY} = -\frac{e^{-Y/\alpha}}{\alpha}.$$

Therefore, by the transformation technique, the probability density function of Y is

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| \\ &= 1 \left| -\frac{e^{-y/\alpha}}{\alpha} \right| \\ &= (1/\alpha)e^{-y/\alpha} \qquad\qquad y > 0, \end{aligned}$$

which is the probability density function of the exponential(α) distribution.

APPL verification: The APPL statements

```
assume(alpha > 0);
X := StandardUniformRV();
g := [[x -> -alpha * ln(x)], [0, infinity]];
Y := Transform(X, g);
```

yield the probability density function of an exponential(α) random variable

$$f_Y(y) = (1/\alpha)e^{-y/\alpha} \qquad\qquad y > 0.$$