Theorem If $X \sim U(0,1)$, then $Y = \lfloor 10^X \rfloor$ has the Benford distribution.

Proof Let $X \sim U(0,1)$ and let $Y = g(X) = \lfloor 10^X \rfloor$. Then,

$$g(x) = \begin{cases} 1 & 0 < x < \log_{10}(2) \\ 2 & \log_{10}(2) < x < \log_{10}(3) \\ 3 & \log_{10}(3) < x < \log_{10}(4) \\ 4 & \log_{10}(4) < x < \log_{10}(5) \\ 5 & \log_{10}(5) < x < \log_{10}(6) \\ 6 & \log_{10}(6) < x < \log_{10}(7) \\ 7 & \log_{10}(7) < x < \log_{10}(8) \\ 8 & \log_{10}(8) < x < \log_{10}(9) \\ 9 & \log_{10}(9) < x < \log_{10}(10) \end{cases}$$

which covers the support of the standard uniform random variable. This transformation yields an associated probability mass function

$$f(y) = \begin{cases} \log_{10}(2) - 0 & y = 1 \\ \log_{10}(3) - \log_{10}(2) & y = 2 \\ \log_{10}(4) - \log_{10}(3) & y = 3 \\ \log_{10}(5) - \log_{10}(4) & y = 4 \\ \log_{10}(6) - \log_{10}(5) & y = 5 \\ \log_{10}(7) - \log_{10}(6) & y = 6 \\ \log_{10}(8) - \log_{10}(7) & y = 7 \\ \log_{10}(9) - \log_{10}(8) & y = 8 \\ \log_{10}(10) - \log_{10}(9) & y = 9, \end{cases}$$

or, more compactly,

$$f(x) = \log_{10}(x+1) - \log_{10}(x) \qquad x = 1, 2, \ldots, 9.$$

This simplifies to the probability mass function of a Benford random variable:

$$f(x) = \log_{10}\left(1 + \frac{1}{x}\right) \qquad x = 1, 2, \ldots, 9.$$