Theorem For $X \sim \text{Pascal}(n, p)$, as $n \to \infty$, $X \sim \text{Poisson}(\lambda)$.

Proof [UNDER CONSTRUCTION!] Let $X \sim \text{Pascal}(n, p)$ with probability mass function

$$f(x) = \frac{(n+x-1)!}{x!(n-1)!} \cdot p^n \cdot (1-p)^x \qquad x = 0, 1, 2, \ldots .$$

Rewrite with $p = \frac{n}{\lambda+n}$, and

$$\begin{aligned} f(x) &= \frac{(n+x-1)!}{x!(n-1)!} \cdot \left(\frac{n}{\lambda+n}\right)^n \cdot \left(1 - \frac{n}{\lambda+n}\right)^x \\ &= \frac{(n+x-1)!}{(n-1)!(\lambda+n)^x} \cdot \left(\frac{\lambda^x}{x!}\right) \cdot \frac{1}{(1+\lambda/n)^n} \qquad x = 0, 1, 2, \ldots . \end{aligned}$$

As $n \to \infty$,

$$\begin{aligned} &\lim_{n\to\infty} \frac{n+x-j}{\lambda+n} \to 1, \qquad j = 1, \ldots, x \\ &\lim_{n\to\infty} \left(1 + \frac{\lambda}{n}\right)^n \to e^{\lambda}. \end{aligned}$$

So the probability mass function in the limit as $n \to \infty$ is

$$\lim_{n\to\infty} f(x) = \lim_{n\to\infty} \frac{(n+x-1)!}{(n-1)!(\lambda+n)^x} \cdot \left(\frac{\lambda^x}{x!}\right) \cdot \frac{1}{(1+\lambda/n)^n} = \frac{\lambda^x}{x!} e^{-\lambda},$$

which is the probability mass function of the Poisson distribution.