Theorem The limiting distribution of a noncentral $\mathrm{F}(n_1, n_2, \delta)$ random variable is $\mathrm{F}(n_1, n_2)$ as $\delta \to 0$.

Proof Let the random variable X have the noncentral $\mathrm{F}(n_1, n_2, \delta)$ distribution with probability density function

$$f_X(x) = \sum_{i=0}^{\infty} \frac{\Gamma\left(\frac{2i+n_1+n_2}{2}\right)\left(\frac{n_1}{n_2}\right)^{(2i+n_1)/2} x^{(2i+n_1-2)/2} e^{-\delta/2}\left(\frac{\delta}{2}\right)^i}{\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{2i+n_1}{2}\right) i! \left(1+\frac{n_1}{n_2}x\right)^{(2i+n_1+n_2)/2}} \qquad x > 0.$$

As $\delta \to 0$, we have

$$\begin{aligned}
\lim_{\delta\to 0} f_X(x) &= \lim_{\delta\to 0} \sum_{i=0}^{\infty} \frac{\Gamma\left(\frac{2i+n_1+n_2}{2}\right)\left(\frac{n_1}{n_2}\right)^{(2i+n_1)/2} x^{(2i+n_1-2)/2} e^{-\delta/2}\left(\frac{\delta}{2}\right)^i}{\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{2i+n_1}{2}\right) i! \left(1+\frac{n_1}{n_2}x\right)^{(2i+n_1+n_2)/2}} \\
&= \lim_{\delta\to 0} \frac{\Gamma\left(\frac{n_1+n_2}{2}\right)\left(\frac{n_1}{n_2}\right)^{n_1/2} x^{(n_1-2)/2} e^{-\delta/2}\left(\frac{\delta}{2}\right)^0}{\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{n_1}{2}\right)\left(1+\frac{n_1}{n_2}x\right)^{(n_1+n_2)/2}} \\
&= \frac{\Gamma\left(\frac{n_1+n_2}{2}\right)\left(\frac{n_1}{n_2}\right)^{n_1/2} x^{(n_1-2)/2} e^{-\delta/2}}{\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{n_1}{2}\right)\left(1+\frac{n_1}{n_2}x\right)^{(n_1+n_2)/2}} \lim_{\delta\to 0}\left(\frac{\delta}{2}\right)^0.
\end{aligned}$$

Now,

$$\lim_{\delta\to 0}\left(\frac{\delta}{2}\right)^0 = 1,$$

so

$$\lim_{\delta\to 0} f_X(x) = \frac{\Gamma\left(\frac{n_1+n_2}{2}\right)\left(\frac{n_1}{n_2}\right)^{n_1/2} x^{(n_1-2)/2}}{\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{n_1}{2}\right)\left(1+\frac{n_1}{n_2}x\right)^{(n_1+n_2)/2}} \qquad x > 0,$$

which is the probability density function of the $\mathrm{F}(n_1, n_2)$ distribution.

APPL verification: The APPL statements

```
X := [[x -> sum((GAMMA((2 * i + n1 + n2) / 2) * (n1 / n2) ^ ((2 * i + n1) / 2)
                * x ^ ((2 * i + n1 - 2) / 2) * exp(-delta / 2) * (delta / 2) ^ i)
                / (GAMMA(n2 / 2) * GAMMA((2 * i + n1) / 2) * i!
                * (1 + (n1 / n2) * x) ^ ((2 * i + n1 + n2) / 2)),
                i = 0 .. infinity)], [0, infinity], ["Continuous", "PDF"]];
limit(X[1][1](x), delta = 0);
```

yield the probability density function of a $\mathrm{F}(n_1, n_2)$ random variable

$$f_Y(y) = \frac{\Gamma\left(\frac{n_1+n_2}{2}\right)\left(\frac{n_1}{n_2}\right)^{n_1/2} y^{(n_1-2)/2}}{\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{n_1}{2}\right)\left(1+\frac{n_1}{n_2}y\right)^{(n_1+n_2)/2}} \qquad y > 0.$$