**Theorem** The limiting distribution of a noncentral beta$(\delta, \beta, \gamma)$ random variable is beta$(\beta, \gamma)$ as $\delta \to 0$.

**Proof** Let the random variable $X$ have the noncentral beta$(\delta, \beta, \gamma)$ distribution with probability density function

$$f_X(x) = \sum_{i=0}^{\infty} \frac{\Gamma(i+\beta+\gamma)}{\Gamma(\gamma)\Gamma(i+\beta)} \left(\frac{e^{\delta/2}}{i!}\right) \left(\frac{\delta}{2}\right)^i x^{i+\beta-1}(1-x)^{\gamma-1} \qquad 0 < x < 1.$$

As $\delta \to 0$, we have

$$\begin{aligned}
\lim_{\delta \to 0} f_X(x) &= \lim_{\delta \to 0} \sum_{i=0}^{\infty} \frac{\Gamma(i+\beta+\gamma)}{\Gamma(\gamma)\Gamma(i+\beta)} \left(\frac{e^{\delta/2}}{i!}\right) \left(\frac{\delta}{2}\right)^i x^{i+\beta-1}(1-x)^{\gamma-1} \\
&= \sum_{i=0}^{\infty} \lim_{\delta \to 0} \left[\frac{\Gamma(i+\beta+\gamma)}{\Gamma(\gamma)\Gamma(i+\beta)} \left(\frac{e^{\delta/2}}{i!}\right) \left(\frac{\delta}{2}\right)^i x^{i+\beta-1}(1-x)^{\gamma-1}\right] \\
&= \lim_{\delta \to 0} \frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)\Gamma(\beta)} e^{\delta/2} \left(\frac{\delta}{2}\right)^0 x^{\beta-1}(1-x)^{\gamma-1} \\
&= \frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)\Gamma(\beta)} x^{\beta-1}(1-x)^{\gamma-1} \lim_{\delta \to 0} \left(\frac{\delta}{2}\right)^0.
\end{aligned}$$

Now,

$$\lim_{\delta \to 0} \left(\frac{\delta}{2}\right)^0 = \lim_{\delta \to 0} 1 = 1.$$

So

$$\lim_{\delta \to 0} f_X(x) = \frac{\Gamma(\gamma+\beta)}{\Gamma(\gamma)\Gamma(\beta)} x^{\beta-1}(1-x)^{\gamma-1} \qquad 0 < x < 1,$$

which is the probability density function of the beta$(\beta, \gamma)$ distribution.

**APPL verification:** The APPL statements

```
X := [[x -> sum((GAMMA(i + b + g) / (GAMMA(g) * GAMMA(i + b))) * (exp(-d / 2) / i!)
          * (d / 2) ^ i * x ^ (i + b - 1) * (1 - x) ^ (g - 1), i = 0 .. infinity)],
          [0, 1], ["Continuous","PDF"]];
limit(X[1][1](x), d=0);
```

yield the probability density function of a beta$(\beta, \gamma)$ random variable

$$f_Y(y) = \frac{\Gamma(\gamma+\beta)}{\Gamma(\gamma)\Gamma(\beta)} y^{\beta-1}(1-y)^{\gamma-1} \qquad 0 < y < 1.$$