Theorem The natural logarithm of a log normal(α, β) random variable is a $N(\mu,\, \sigma^2)$ random variable.

Proof Let the random variable X have the log normal distribution with probability density function

$$f_X(x) = \frac{1}{\sqrt{2\pi} x\beta}\, e^{-\frac{1}{2}\left(\frac{\ln(x/\alpha)}{\beta}\right)^2} \qquad x > 0.$$

The transformation $Y = g(X) = \ln(X)$ is a 1–1 transformation from $\mathcal{X} = \{x \,|\, x > 0\}$ to $\mathcal{Y} = \{y \,|\, -\infty < y < \infty\}$ with inverse $X = g^{-1}(Y) = e^Y$ and Jacobian

$$\frac{dX}{dY} = e^Y.$$

Therefore, by the transformation technique, the probability density function of Y is

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \left|\frac{dx}{dy}\right| \\ &= \frac{1}{\sqrt{2\pi} e^y \beta}\, e^{-\frac{1}{2}\left(\frac{\ln(e^y/\alpha)}{\beta}\right)^2} |e^y| \\ &= \frac{1}{\sqrt{2\pi}\beta}\, e^{-\frac{1}{2}\left(\frac{y-\ln(\alpha)}{\beta}\right)^2} \qquad -\infty < y < \infty. \end{aligned}$$

Let $\alpha = e^\mu$ and $\beta = \sigma$. Then

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma}\, e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} \qquad -\infty < y < \infty,$$

which is the probability density function of the normal distribution.

APPL verification: The APPL statements

```
X := LogNormalRV(mu, sigma);
g := [[x -> ln(x)], [0, infinity]];
Y := Transform(X, g);
Z := NormalRV(mu, sigma);
```

yield identical functional forms

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma}\, e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} \qquad -\infty < y < \infty$$

for the random variables Y and Z, which verifies that the natural logarithm of a log normal random variable has the normal distribution.