Theorem If $X_i \sim$ exponential(λ_i) for $i = 1, 2, \ldots, n$ are mutually independent random variables with survivor functions

$$S_i(x) = e^{-\lambda_i x} \qquad x > 0,$$

for $i = 1, 2, \ldots, n$ and $\lambda_i \neq \lambda_j$ for $i \neq j$, then $X_1 + X_2 + \cdots + X_n$ has the hypoexponential probability density function

$$f_{X_1+X_2+\cdots+X_n}(x) = \sum_{i=1}^{n} \lambda_i e^{-\lambda_i x} \prod_{j=1, j\neq i}^{n} \left(\frac{\lambda_j}{\lambda_j - \lambda_i}\right) \qquad x > 0.$$

Proof Given in Ross (2007) *Introduction to Probability Models*, 8th ed., Academic Press by using induction by first showing the case of $n = 2$, then that showing that case n implies case $n + 1$.

Illustration. We show the case of $n = 2$ here.

Option 1. Use the cumulative distribution function technique.

$$\begin{aligned} F_{X_1+X_2}(x) &= P(X_1 + X_2 \leq x) \\ &= \int_0^x \int_0^{x-x_1} \lambda_1 e^{-\lambda_1 x_1} \lambda_2 e^{-\lambda_2 x_2} dx_2\, dx_1 \\ &= \cdots \\ &= 1 - \left(\frac{\lambda_1}{\lambda_1 - \lambda_2}\right) e^{-\lambda_2 x} - \left(\frac{\lambda_2}{\lambda_2 - \lambda_1}\right) e^{-\lambda_1 x} \qquad x > 0. \end{aligned}$$

So the probability density function of $X_1 + X_2$ is

$$f_{X_1+X_2}(x) = \left(\frac{\lambda_2}{\lambda_2 - \lambda_1}\right) \lambda_1 e^{-\lambda_1 x} + \left(\frac{\lambda_1}{\lambda_1 - \lambda_2}\right) \lambda_2 e^{-\lambda_2 x} \qquad x > 0.$$

Option 2. Use the convolution formula. Since X_1 and X_2 are independent,

$$\begin{aligned} f_{X_1+X_2}(x) &= \int_0^x f_{X_1}(t) f_{X_2}(x-t) dt \\ &= \int_0^x \lambda_1 e^{-\lambda_1 t} \lambda_2 e^{-\lambda_2 (x-t)} dt \\ &= \cdots \\ &= \left(\frac{\lambda_2}{\lambda_2 - \lambda_1}\right) \lambda_1 e^{-\lambda_1 x} + \left(\frac{\lambda_1}{\lambda_1 - \lambda_2}\right) \lambda_2 e^{-\lambda_2 x} \qquad x > 0. \end{aligned}$$

Option 3. Use the transformation technique. (Tedious due to the dummy transformation.)

APPL illustration: For $n = 3$, the statement

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Convolution(ExponentialRV(lam1), ExponentialRV(lam2), ExponentialRV(lam3))
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gives

$$f_X(x) = \left(\frac{\lambda_2}{\lambda_2 - \lambda_1}\right)\left(\frac{\lambda_3}{\lambda_3 - \lambda_1}\right) \lambda_1 e^{-\lambda_1 x} + \left(\frac{\lambda_1}{\lambda_1 - \lambda_2}\right)\left(\frac{\lambda_3}{\lambda_3 - \lambda_2}\right) \lambda_2 e^{-\lambda_2 x} + \left(\frac{\lambda_1}{\lambda_1 - \lambda_3}\right)\left(\frac{\lambda_2}{\lambda_2 - \lambda_3}\right) \lambda_3 e^{-\lambda_3 x} \qquad x > 0.$$