Theorem [UNDER CONSTRUCTION!] The limiting distribution of a doubly noncentral $F(n_1, n_2, \delta, \gamma)$ random variable is noncentral $F(n_1, n_2, \delta)$ as $\gamma \to 0$.

Proof [UNDER CONSTRUCTION!] Let the random variable X have the doubly noncentral $F(n_1, n_2, \delta, \gamma)$ distribution with probability density function

$$f(x) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left[\frac{e^{-\delta/2}(\delta/2)^j}{j!}\right]\left[\frac{e^{-\gamma/2}(\gamma/2)^k}{k!}\right] n_1^{n_1/2+j} n_2^{n_2/2+k} x^{n_1/2+j-1}$$

$$\times (n_2 + n_1 x)^{-(n_1+n_2)/2-j-k}\left[B\left(n_1/2+j, n_2/2+k\right)\right]^{-1} \qquad x > 0.$$

For all nonzero k, $\lim_{\gamma\to 0} f(x) = 0$. So consider only the case of $k = 0$. As $\gamma \to 0$,

$$\begin{aligned}
\lim_{\gamma\to 0} f(x) &= \lim_{\gamma\to 0}\sum_{j=0}^{\infty}\left[\frac{e^{-\delta/2}\left(\frac{\delta}{2}\right)^j}{j!}\right]\left[e^{-\gamma/2}\left(\frac{\gamma}{2}\right)^0\right] n_1^{(n_1/2)+j} n_2^{(n_2/2)} x^{(n_1/2)+j-1} \\
&\quad \times (n_2 + n_1 x)^{-\frac{1}{2}(n_1+n_2)-j}\left[B\left(n_1/2+j, n_2/2\right)\right]^{-1} \\
&= \left[\sum_{j=0}^{\infty}\frac{\Gamma\left(\frac{2j+n_1+n_2}{2}\right)\left(\frac{n_1}{n_2}\right)^{(2j+n_1)/2} x^{(2j+n_1-2)/2} e^{-\delta/2}\left(\frac{\delta}{2}\right)^j}{\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{2j+n_1}{2}\right) j!\left(1+\frac{n_1}{n_2}x\right)^{(2j+n_1+n_2)/2}}\right]\cdot \lim_{\gamma\to 0}\left(\frac{\gamma}{2}\right)^0 \qquad x > 0.
\end{aligned}$$

Using L'Hopital's Rule repeatedly,

$$\begin{aligned}
\lim_{\gamma\to 0}\left(\frac{\gamma}{2}\right)^0 &= \lim_{\gamma\to 0} e^{0\cdot\ln\left(\frac{\gamma}{2}\right)} \\
&= e^{\lim_{\gamma\to 0} 0\cdot\ln\left(\frac{\gamma}{2}\right)} \\
&= e^{\lim_{\gamma\to 0}\frac{\ln\left(\frac{\gamma}{2}\right)}{\frac{1}{0}}} \\
&= e^{\lim_{\gamma\to 0}\frac{2/\gamma}{1/0}} \\
&= e^{\lim_{\gamma\to 0}\frac{0}{\gamma}} \\
&= e^{\lim_{\gamma\to 0}\frac{0}{1}} \\
&= e^0 \\
&= 1.
\end{aligned}$$

So,

$$\lim_{\gamma\to 0} f(x) = \sum_{i=0}^{\infty}\frac{\Gamma\left(\frac{2i+n_1+n_2}{2}\right)\left(\frac{n_1}{n_2}\right)^{(2i+n_1)/2} x^{(2i+n_1-2)/2} e^{-\delta/2}\left(\frac{\delta}{2}\right)^i}{\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{2i+n_1}{2}\right) i!\left(1+\frac{n_1}{n_2}x\right)^{(2i+n_1+n_2)/2}} \qquad x > 0,$$

which is the probability density function of the noncentral $F(n_1, n_2, \delta)$ distribution.

APPL verification: The APPL statements

```
X := [[x -> sum(sum(((exp(-d / 2) * (d / 2) ^ j) / j!) *
                   ((exp(-g / 2) * (g / 2) ^ k) / k!) *
                   n1 ^ ((n1 / 2) + j) * n2 ^ ((n2 / 2) + k) *
                   x ^ ((n1 / 2) + j - 1) * (n2 + n1 * x) ^
                   (-(n1 + n2) / 2 - j - k) /
                   BETA(n1 / 2 + j, n2 / 2 + k),
                   k = 0 .. infinity), j = 0 .. infinity)],
      [0, infinity], ["Continuous", "PDF"]];
PDFY := limit(X[1][1](x), g = 0);
```

yield the probability density function of a noncentral $F(n_1, n_2, \delta)$ random variable

$$f(x) = \sum_{i=0}^{\infty} \frac{\Gamma\left(\frac{2i+n_1+n_2}{2}\right)\left(\frac{n_1}{n_2}\right)^{(2i+n_1)/2} x^{(2i+n_1-2)/2} e^{-\delta/2}\left(\frac{\delta}{2}\right)^i}{\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{2i+n_1}{2}\right) i! \left(1+\frac{n_1}{n_2}x\right)^{(2i+n_1+n_2)/2}} \qquad x > 0.$$