Budget Variance Analysis of $n$-Variable Products with Zero or $n$ Responsibility Centers

ABSTRACT
This note aims to prove the mathematics behind a new set of budget variance analysis models, specifically, those for $n$–variable products with zero or $n$ responsibility centers. We demonstrate that the associated differences of products can be expressed as a function of averages and differences of individual values. The models can be used in all Business disciplines currently utilizing variance analysis, such as Accounting, Economics, Finance, Operations Management and Marketing.

Keywords: Budget analysis, Combinations, Cost accounting, Financial analysis, Horizontal analysis, Management accounting, Margin drivers, Performance management, Responsibility centers.
I. INTRODUCTION

Corresponding Author (2023) proposes a new set of variance analysis models for \( n \)-variable products based on the concept of responsibility centers (i.e., accountable decision-makers or departments). For an \( n \)-variable product, there can be 0, 1, 2, . . . , \( n \) responsibility centers and the models differ based on both the number and combination of responsibility centers. Here, we show that for models of an \( n \)-variable product with \( n \)-responsibility centers, the variance of each variable can be expressed as simply as a function of averages and differences of individual values. Corresponding Author (2023) shows that for a given \( n \), the corresponding zero and \( n \)-responsibility center models are the same. Therefore, the results derived here for the \( n \)-responsibility center models also apply to the corresponding zero-responsibility center models.

To begin, consider the \( n = 2 \) revenue case. Let \( p \) be the unit price of a product and \( q \) be the quantity sold of that product. Furthermore, let

- \( p_b \) be the budgeted unit price,
- \( p_a \) be the actual unit price,
- \( q_b \) be the budgeted quantity sold, and
- \( q_a \) be the actual quantity sold.

Define the difference of unit price and the difference of quantity sold as

\[
\Delta p = p_a - p_b \quad \text{and} \quad \Delta q = q_a - q_b.
\]

Define the average unit price and the average quantity sold as

\[
\bar{p} = \frac{p_b + p_a}{2} \quad \text{and} \quad \bar{q} = \frac{q_b + q_a}{2}.
\]
The expression

\[ p_a q_a - p_b q_b \]

is the difference of actual and budgeted revenues. Five key observations concerning this formula are given next. First, the most commonly-found variance analysis model for this expression is

\[ p_a q_a - p_b q_b = \Delta p q_a + \Delta q p_b \] (1)

where the first term on the right-hand side of the equation is the portion of the difference in revenue corresponding to the difference in unit price (i.e., price variance) and the second term is the portion of the difference in revenue corresponding to the difference of quantity sold (i.e., quantity variance). Even though the model is ubiquitous in textbooks and practice, criticisms include “. . . the textbook example is solved by arbitrarily adding the joint variance to the price variance. There is no theoretical justification for so doing.” (Kloock, J. & Schiller, U., 1997), and “. . . the conventional two-variance analysis (price and quantity) inflates variances in three of the four possible economic situations.” (Mitchell, T. & Thomas, M., 2005). Such comments lead to an obvious question, “Why is Equation 1 used, as opposed to

\[ p_a q_a - p_b q_b = \Delta p q_b + \Delta q p_a ? \] (2)

To investigate, Corresponding Author (2023) uses a simple numerical example (reproduced in Appendix A) to demonstrate that Model 1 is biased in favor of price variance, and Model 2 is biased in favor of quantity variance. The same conclusion can be shown for variance analysis of higher-order products, such as three-variable spending. This observation alone suggests the need for justification of one model over the other, or possibly the need for a new model altogether to be used under certain circumstances (see Model 3 below). The latter is the focus of this note. In addition, the example demonstrates the non-trivial differences between price (and quantity) variance that can arise when comparing the results of the two models. While such drastic differences may be rare for a mature firm with little actual
price and/or sales deviations from budget, for rapidly-growing firms this may be the norm. Second, Corresponding Author (2023) incorporates the concept of responsibility centers and proposes 1) Model 1 is appropriate for a firm with a decision maker responsible for unit price, but there is no decision maker responsible for quantity sold (e.g., a firm that sets the price for its product and makes it available for sale online with no further sales effort), 2) Model 2 is appropriate for a firm with a decision maker responsible for quantity sold, but there is no decision maker responsible for setting unit price (e.g., a firm that has an active sales force responsible for selling a pure commodity at a spot price determined by the market), and 3) for a firm with decision makers accountable for unit price and quantity sold (e.g., a cartel that can both ration units sold in the marketplace and set the selling price) the difference of revenues can be partitioned as

\[ p_aq_a - p_bq_b = \Delta p\bar{q} + \Delta q\bar{p}. \] (3)

As with Models 1 and 2, the first term on the right-hand side of the equation is the price variance and the second term is the quantity variance. Third, Corresponding Author (2023) illustrates the application of a generalization of this formula to the case of more than two factors (see the three-variable direct materials spending example presented in Section III). Fourth, the application of this formula is not limited to just variance analysis. It can also be applied to two-period horizontal analysis, for example, comparing the revenues from two time periods. A example is a four-factor planning model which appears in Marketing textbooks (Spiro et al., 2003). Finally, the models discussed in this note and the algorithm used to generate them are not limited to just accounting applications. The algorithm can be used to generate a solution to the well-known Bankruptcy Problem in game theory (Aumann and Maschler, 1985) with \( n \) creditors collectively having a sum of claims greater than the value of the bankrupt firm. Another obvious game-theoretic application is executive compensation. Consider \( n \) executives discussing \textit{ex ante} how to assign credit or blame should a revenue or
spending variance occur. The models discussed in this note are neutral and can be agreed upon in advance to calculate unbiased variances after actual results are recognized.

The focus here is to state and prove a theorem that concerns the calculation of differences of certain products.

II. RESULT

The theorem below generalizes the example in the introduction to apply to more than just the $n = 2$ variables unit price and quantity sold. The theorem considers $n$ variables that can each assume two values. The result indicates that the difference of the products can be written as a function of the averages and differences between the two values.

**Theorem 1.** Consider the variables $x_i$, for $i = 1, 2, \ldots, n$, that can each only assume the two values $x_{i,1}$ and $x_{i,2}$, for $i = 1, 2, \ldots, n$. Define the difference of $x_i$ as

$$\Delta x_i = x_{i,2} - x_{i,1}$$

and the average of $x_i$ as

$$\bar{x}_i = \frac{x_{i,1} + x_{i,2}}{2}$$

for $i = 1, 2, \ldots, n$. The difference of the products can be expressed as

$$\prod_{i=1}^{n} x_{i,2} - \prod_{i=1}^{n} x_{i,1} = \sum_{d=1, d \text{ odd}}^{n} \frac{1}{2d-1} \sum_{S \subseteq [n], |S|=d} \prod_{j \in S, k \in S'} \Delta x_j \bar{x}_k,$$  \hspace{1cm} (1)

where $[n] = \{1, 2, \ldots, n\}$, $S$ is any subset of $[n]$, and $S'$ is the complement of $S$. The terms in the implementation of the right-hand sides of Equation 1 for $n = 2, 3, \ldots, 6$ are given in Table 1.
Table 1
Terms on the Right-Hand Sides of Equation 1.

<table>
<thead>
<tr>
<th>$n = 2$</th>
<th>$n = 3$</th>
<th>$n = 4$</th>
<th>$n = 5$</th>
<th>$n = 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta x_1 \bar{x}_2$</td>
<td>$\Delta x_1 \bar{x}_2 \bar{x}_3$</td>
<td>$\Delta x_1 \bar{x}_2 \bar{x}_3 \bar{x}_4$</td>
<td>$\Delta x_1 \bar{x}_2 \bar{x}_3 \bar{x}_4 \bar{x}_5$</td>
<td>$\Delta x_1 \bar{x}_2 \bar{x}_3 \bar{x}_4 \bar{x}_5 \bar{x}_6$</td>
</tr>
<tr>
<td>$\bar{x}_1 \Delta x_2$</td>
<td>$\bar{x}_1 \Delta x_2 \bar{x}_3$</td>
<td>$\bar{x}_1 \Delta x_2 \bar{x}_3 \bar{x}_4$</td>
<td>$\bar{x}_1 \Delta x_2 \bar{x}_3 \bar{x}_4 \bar{x}_5$</td>
<td>$\bar{x}_1 \Delta x_2 \bar{x}_3 \bar{x}_4 \bar{x}_5 \bar{x}_6$</td>
</tr>
<tr>
<td>$\bar{x}_1 \bar{x}_2 \Delta x_3$</td>
<td>$\bar{x}_1 \bar{x}_2 \Delta x_3 \bar{x}_4$</td>
<td>$\bar{x}_1 \bar{x}_2 \Delta x_3 \bar{x}_4 \bar{x}_5$</td>
<td>$\bar{x}_1 \bar{x}_2 \Delta x_3 \bar{x}_4 \bar{x}_5 \bar{x}_6$</td>
<td>$\bar{x}_1 \bar{x}_2 \Delta x_3 \bar{x}_4 \bar{x}_5 \bar{x}_6$</td>
</tr>
<tr>
<td>$\Delta x_1 \Delta x_2 \Delta x_3 / 4$</td>
<td>$\Delta x_1 \Delta x_2 \Delta x_3 \Delta x_4 / 4$</td>
<td>$\Delta x_1 \Delta x_2 \Delta x_3 \Delta x_4 \Delta x_5 / 4$</td>
<td>$\Delta x_1 \Delta x_2 \Delta x_3 \Delta x_4 \Delta x_5 \Delta x_6 / 4$</td>
<td>$\Delta x_1 \Delta x_2 \Delta x_3 \Delta x_4 \Delta x_5 \Delta x_6 / 4$</td>
</tr>
<tr>
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<td>$\Delta x_1 \Delta x_2 \Delta x_3 \Delta x_4 \Delta x_5 / 4$</td>
<td>$\Delta x_1 \Delta x_2 \Delta x_3 \Delta x_4 \Delta x_5 \Delta x_6 / 4$</td>
<td>$\Delta x_1 \Delta x_2 \Delta x_3 \Delta x_4 \Delta x_5 \Delta x_6 / 4$</td>
<td>$\Delta x_1 \Delta x_2 \Delta x_3 \Delta x_4 \Delta x_5 \Delta x_6 / 4$</td>
</tr>
<tr>
<td>$\Delta x_1 \Delta x_2 \Delta x_3 \Delta x_4 \Delta x_5 / 4$</td>
<td>$\Delta x_1 \Delta x_2 \Delta x_3 \Delta x_4 \Delta x_5 \Delta x_6 / 4$</td>
<td>$\Delta x_1 \Delta x_2 \Delta x_3 \Delta x_4 \Delta x_5 \Delta x_6 / 4$</td>
<td>$\Delta x_1 \Delta x_2 \Delta x_3 \Delta x_4 \Delta x_5 \Delta x_6 / 4$</td>
<td>$\Delta x_1 \Delta x_2 \Delta x_3 \Delta x_4 \Delta x_5 \Delta x_6 / 4$</td>
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<tr>
<td>$\Delta x_1 \Delta x_2 \Delta x_3 \Delta x_4 \Delta x_5 \Delta x_6 / 4$</td>
<td>$\Delta x_1 \Delta x_2 \Delta x_3 \Delta x_4 \Delta x_5 \Delta x_6 / 4$</td>
<td>$\Delta x_1 \Delta x_2 \Delta x_3 \Delta x_4 \Delta x_5 \Delta x_6 / 4$</td>
<td>$\Delta x_1 \Delta x_2 \Delta x_3 \Delta x_4 \Delta x_5 \Delta x_6 / 4$</td>
<td>$\Delta x_1 \Delta x_2 \Delta x_3 \Delta x_4 \Delta x_5 \Delta x_6 / 4$</td>
</tr>
</tbody>
</table>

### III. DISCUSSION

A formal proof of Theorem 1 is given in the appendix. We give the motivation associated with the proof here. The $1/2^{d-1}$ factor in the result is used to account for the 2 in the
denominator of the averages. So temporarily writing \( x_{i,1} \) as \( b_i \) and \( x_{i,2} \) as \( a_i \), we need to show that

\[
\prod_{i=1}^{n} a_i - \prod_{i=1}^{n} b_i
\]

is a function of the averages and differences of the \( b_i \) and \( a_i \) values. The choice of the variables’ names \( a_i \) and \( b_i \) is consistent with their interpretation as actual and budgeted values from variance analysis. When \( n = 4 \), for example, we want to show that the difference of the products is

\[
a_1a_2a_3a_4 - b_1b_2b_3b_4 = \frac{1}{8} \left[ (a_1 - b_1)(a_2 + b_2)(a_3 + b_3)(a_4 + b_4) + (a_1 + b_1)(a_2 - b_2)(a_3 + b_3)(a_4 + b_4) + 
(a_1 + b_1)(a_2 + b_2)(a_3 - b_3)(a_4 + b_4) + 
(a_1 + b_1)(a_2 + b_2)(a_3 + b_3)(a_4 - b_4) + 
(a_1 - b_1)(a_2 - b_2)(a_3 - b_3)(a_4 + b_4) + 
(a_1 - b_1)(a_2 - b_2)(a_3 + b_3)(a_4 - b_4) + 
(a_1 - b_1)(a_2 + b_2)(a_3 - b_3)(a_4 - b_4) + 
(a_1 + b_1)(a_2 - b_2)(a_3 - b_3)(a_4 - b_4) \right].
\]

The first \( \binom{4}{1} = 4 \) terms on the right-hand side of this equation have a single ‘−’ and the next \( \binom{4}{3} = 4 \) terms on the right-hand side of this equation have three ‘−’s in the terms. The key to the proof is to see that all of the terms on the right-hand side of this equation cancel except for the monomials \( a_1a_2a_3a_4 \) and \( b_1b_2b_3b_4 \).

As a particular example, consider direct materials spending with \( n = 3 \) variables: unit cost \( c = x_1 \), usage \( u = x_2 \), and quantity \( q = x_3 \). Refer to the \( n = 3 \) column in Table 1. Dividing the term \( \Delta x_1 \Delta x_2 \Delta x_3 / 4 \) equally among the three factors (since each of the three factors appears as its respective difference in the term), an analyst performing variance
analysis could assign the difference of spending to unit cost as $\Delta c\bar{u}\bar{q} + \Delta c\Delta u\Delta q/12$, to usage as $\bar{c}\Delta u\bar{q} + \Delta c\Delta u\Delta q/12$, and to quantity as $\bar{c}\bar{u}\Delta q + \Delta c\Delta u\Delta q/12$. These three partitions correspond to those derived in Corresponding Author (2023) for a three-variable product with zero or three responsibility centers.

IV. SUMMARY

We demonstrated how the existing two-variable revenue variance analysis model is biased in favor of price variance and discussed the opportunity for improvement by incorporating the concept of responsibility centers. We proved that the difference of a product of $n$ variables can be expressed as simply as a function of averages and differences. This result supports a newly-proposed set of variance analysis models for an $n$-variable product with zero or $n$ responsibility centers.

V. REFERENCES


A. APPENDIX: NUMERICAL EXAMPLE

Here, we reproduce the two-variable revenue numerical example from Corresponding Author (2023). Where applicable, the marketing department is responsible for unit price, and the sales department is responsible for quantity sold. Model 1A corresponds to Equation 1 and is the proposed model for when the marketing department is the only responsibility center. Model 1B corresponds to Equation 2 and is the proposed model for when the sales department is the only responsibility center. Model 1C corresponds to Equation 3 and is the proposed model for when both the marketing department and sales department are responsibility centers.

Table 2
Variance Models for Two-Variable Revenue

<table>
<thead>
<tr>
<th>Variable</th>
<th>Responsibility Center(s)</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Marketing 1A</td>
<td>Sales 1B</td>
<td>Marketing and Sales* 1C</td>
</tr>
<tr>
<td>Unit Price</td>
<td>$\Delta pq_a$</td>
<td>$\Delta pq_b$</td>
<td>$\Delta p\bar{q}$</td>
</tr>
<tr>
<td>Quantity</td>
<td>$\Delta qp_b$</td>
<td>$\Delta qp_a$</td>
<td>$\Delta q\bar{p}$</td>
</tr>
<tr>
<td>Revenue</td>
<td>$p_a q_a - p_b q_b$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Also applicable for the case of no responsibility centers.

Table 3
Parameters for Two-Variable Revenue Example

<table>
<thead>
<tr>
<th>Variable</th>
<th>Case 1</th>
<th>Case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Budgeted</td>
<td>Actual</td>
</tr>
<tr>
<td>Unit Price</td>
<td>$10</td>
<td>$11</td>
</tr>
<tr>
<td>Quantity</td>
<td>100</td>
<td>200</td>
</tr>
</tbody>
</table>
### Table 4
Results of Two-Variable Revenue Example

<table>
<thead>
<tr>
<th>Variable</th>
<th>Case 1</th>
<th>Case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Marketing</td>
<td>Sales</td>
</tr>
<tr>
<td></td>
<td>1A</td>
<td>1B</td>
</tr>
<tr>
<td>Unit Price</td>
<td>$200</td>
<td>$100</td>
</tr>
<tr>
<td>Quantity</td>
<td>$1000</td>
<td>$1100</td>
</tr>
<tr>
<td>Revenue</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Also applicable for the case of no responsibility centers.

### B. APPENDIX: PROOF OF THEOREM 1

For the purposes of this proof, let $x_{i,1} = b_i$ and $x_{i,2} = a_i$, for $i = 1, 2, \ldots, n$. It is sufficient to show that

$$\prod_{i=1}^{n} a_i - \prod_{i=1}^{n} b_i = \frac{1}{2^{n-1}} \sum_{d=1, \text{odd}}^{n} \sum_{S \subseteq [n], |S| = d} \prod_{j \in S, k \in S'} (a_j - b_j)(a_k + b_k)$$  \hspace{1cm} (2)$$

so that the right-hand side of the equation will be written entirely in terms of sums and differences. The result is proven by showing that of all of the monomials resulting by multi-
plying out the right-hand side of this equation, for example, $a_1a_2b_3a_4\ldots b_n$, all terms cancel except for $\prod_{i=1}^{n} a_i$ and $\prod_{i=1}^{n} b_i$. For the terms to cancel, there must be an equal number of positive and negative terms comprising the monomial. The number of odd-order subsets of $[n]$ is $2^{n-1}$. This can be seen by expanding $(1 - 1)^n$ by the binomial theorem:

$$
(1 - 1)^n = \sum_{i=0}^{n} (-1)^n \binom{n}{i} = 0.
$$

Separating the negative and positive terms,

$$
\sum_{i=0, i \text{ odd}}^{n} \binom{n}{i} = \sum_{i=0, i \text{ even}}^{n} \binom{n}{i}.
$$

Since

$$
\sum_{i=0}^{n} \binom{n}{i} = 2^n,
$$

there are $2^{n-1}$ odd-ordered subsets and $2^{n-1}$ even-ordered subsets of $[n]$.

Now consider an arbitrary monomial resulting in multiplying out the terms on the right-hand side of Equation (2). To show that all terms except $\prod_{i=1}^{n} a_i$ and $\prod_{i=1}^{n} b_i$ cancel, consider the following cases involving a particular arbitrary monomial other than $\prod_{i=1}^{n} a_i$ or $\prod_{i=1}^{n} b_i$.

**Case 1.** The number of times that the monomial is *negative* equals the number of odd-order subsets of $[n]$ containing an *odd* number of $k$-element index sets, which equals the product of the number of *odd*-order subsets of $k$-element sets and the number of *even*-order subsets of $(n - k)$-element complement sets, which, by the multiplication rule, is

$$
2^{k-1} \cdot 2^{n-k-1} = 2^{n-2}.
$$

**Case 2.** The number of times that the monomial is *positive* equals the number of odd-order subsets of $[n]$ containing an *even* number of $k$-element index sets, which equals the product of the number of *even*-order subsets of $k$-element sets and the number of *odd*-order subsets of $(n - k)$-element complement sets, which, by the multiplication rule, is

$$
2^{k-1} \cdot 2^{n-k-1} = 2^{n-2}.
$$
subsets of \((n - k)\)-element complement sets, which, by the multiplication rule, is

\[2^{k-1} \cdot 2^{n-k-1} = 2^{n-2}.\]

Since there are an equal number of positive and negative terms on any monomial except 
\(\prod_{i=1}^{n} a_i\) or \(\prod_{i=1}^{n} b_i\), they must cancel. All of the \(2^{n-1}\) products involving the monomial 
\(a_1 a_2 \ldots a_n\) are positive and all of the \(2^{n-1}\) products involving the monomial \(b_1 b_2 \ldots b_n\) are negative, which proves the result.