IIE Transactions
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713772245

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Online publication date: 02 February 2010

To cite this Article Leemis, Lawrence M.(2010) 'Reliability growth via testing', IIE Transactions, 42: 4, 317 — 324
To link to this Article: DOI: 10.1080/07408170903398000
URL: http://dx.doi.org/10.1080/07408170903398000

IIE Transactions (2010) **42**, 317–324

ISSN: 0740-817X print / 1545-8830 online
DOI: 10.1080/07408170903398000

Reliability growth via testing

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Received April 2008 and accepted January 2009

Observed data values are typically assumed to come from an infinite population of items in reliability and survival analysis applications. The case of a finite population of items with exponentially distributed lifetimes is considered here. The data set consists of the lifetimes of a large number of items that are known to have exponentially distributed failure times with a failure rate that is known with high precision. Failure of the items is not self-announcing, as is the case with a smoke detector. A significant fraction of the items are sampled periodically, and the items that have failed are repaired to a like-new condition with respect to their survival distribution. The goal is to assess the impact of this periodic sampling and repair on the overall finite population reliability over time.

Keywords: Exponential distribution, finite population, reliability estimation, survivor function

1. Introduction

Consider a finite population of n identical repairable items that operate continuously until failure. We assume that each item has an exponential (λ) time to failure, where the failure rate $\lambda > 0$ is assumed to be a known fixed constant. This assumption is based on a long history of testing and the fact that the items consist mainly of electrical components arranged in series. If there is no intervention (e.g., periodic testing and repair) in the finite population of items, the population survivor function $S(t)$ is (Ross, 2007, p. 282):

$$S(t) = e^{-\lambda t}, \qquad t > 0.$$

Assume that failure can only be detected by testing. At ordered times $t_1, t_2, \ldots, t_k$, select $n_1, n_2, \ldots, n_k$ items at random and without replacement at each sampling time from the finite population, and observe $Y_1, Y_2, \ldots, Y_k$ of these items that pass the test. Testing is instantaneous. Any item that fails a test undergoes a perfect repair with no time delay, and is immediately returned to the population with those that pass the test. We consider the estimation of the population survivor function at time t under this periodic sampling scheme by presenting a series of models of increasing complexity by progressively relaxing conditions in order to arrive at a general model for survivor function estimation for finite populations.

2. Models

The goal of each of the four models presented in this section is to give the survivor function for a finite population of items, each with exponential (λ) times to failure. The first model considers a single ($k = 1$) sampling time. The second model considers two ($k = 2$) sampling times. The third model considers the general case of k sampling times. Finally, the fourth model considers the case of a varying population size.

2.1. *Single sampling time*

Assume that there is only a single ($k = 1$) sampling time. The number of passing items Y_1 (out of the n_1 items sampled at time t_1) has the hypergeometric distribution with conditional probability mass function:

$$f_{Y_1 \mid G_1 = g_1}(y \mid G_1 = g_1) = \frac{\binom{g_1}{y}\binom{n-g_1}{n_1-y}}{\binom{n}{n_1}}, \quad y = 0, 1, 2, \ldots, n_1.$$

This probability mass function is conditioned on $G_1 = g_1$ good items and $n - g_1$ bad items in the population at time t_1. Note that

$$\binom{r}{s} = 0,$$

by convention, whenever $s < 0$ or $s > r$, which allows the support of the random variable Y_1 to be written in the simple fashion presented above. The random variable G_1 has the binomial distribution with parameters n and $e^{-\lambda t_1}$, and probability mass function:

$$f_{G_1}(g) = \binom{n}{g}(e^{-\lambda t_1})^g (1 - e^{-\lambda t_1})^{n-g}, \quad g = 0, 1, 2, \ldots, n.$$

Hence, the number of items Y_1 that pass the test at time t_1 has a "binomial–hypergeometric" distribution with unconditional probability mass function:

$$f_{Y_1}(y) = \sum_{g=0}^{n} f_{Y_1 \mid G_1=g}(y \mid G_1 = g) f_{G_1}(g)$$
$$= \sum_{g=0}^{n} \frac{\binom{g}{y}\binom{n-g}{n_1-y}}{\binom{n}{n_1}} \binom{n}{g} (e^{-\lambda t_1})^g (1 - e^{-\lambda t_1})^{n-g},$$
$$y = 0, 1, 2, \ldots, n_1.$$

Example 1. Assume that: (i) the single sampling time occurs at time $t_1 = 1$; (ii) the population failure rate is $\lambda = -\ln(2/3)$; (iii) there are $n = 5$ items in the population; and (iv) there are $n_1 = 3$ items sampled at time $t_1 = 1$.

In this case, the conditional probability mass function of Y_1 is

$$f_{Y_1 \mid G_1=g_1}(y \mid G_1 = g_1) = \frac{\binom{g_1}{y}\binom{5-g_1}{3-y}}{\binom{5}{3}}, \quad y = 0, 1, 2, 3,$$

for $g_1 = 0, 1, 2, 3, 4, 5$. The probability mass function of G_1 is

$$f_{G_1}(g) = \binom{5}{g}\left(\frac{2}{3}\right)^g\left(\frac{1}{3}\right)^{5-g}, \quad g = 0, 1, 2, 3, 4, 5.$$

The unconditional probability mass function of the number of passing items Y_1 is

$$f_{Y_1}(y) = \sum_{g=0}^{5} \frac{\binom{g}{y}\binom{5-g}{3-y}}{\binom{5}{3}} \binom{5}{g}\left(\frac{2}{3}\right)^g\left(\frac{1}{3}\right)^{5-g}, \quad y = 0, 1, 2, 3,$$

which simplifies to

$$f_{Y_1}(y) = \begin{cases} 1/27 & \text{for} \quad y = 0, \\ 2/9 & \text{for} \quad y = 1, \\ 4/9 & \text{for} \quad y = 2, \\ 8/27 & \text{for} \quad y = 3. \end{cases}$$

Now consider the effect of the sampling, testing, and possible repairs that occur at time t_1 on the population survivor function at time t, which is denoted by $S(t)$. Prior to time t_1, the survivor function is

$$S(t) = e^{-\lambda t}, \quad 0 < t \le t_1.$$

After the test that occurs at time t_1, the population is a finite mixture of the tested group of n_1 items and the untested group of $n - n_1$ items, so the population survivor function is

$$S(t) = \frac{n - n_1}{n} e^{-\lambda t} + \frac{n_1}{n} e^{-\lambda(t-t_1)}, \quad t > t_1.$$

The $(t - t_1)$ term accounts for the renewal of the n_1 items that were tested at time t_1. Those items that were found to be failed are assumed to be repaired to a good-as-new condition; those items that were found to be operating are as good as new because of the memoryless property associated with the exponential distribution. The testing results in a discontinuous increase in $S(t)$ at the testing time t_1.

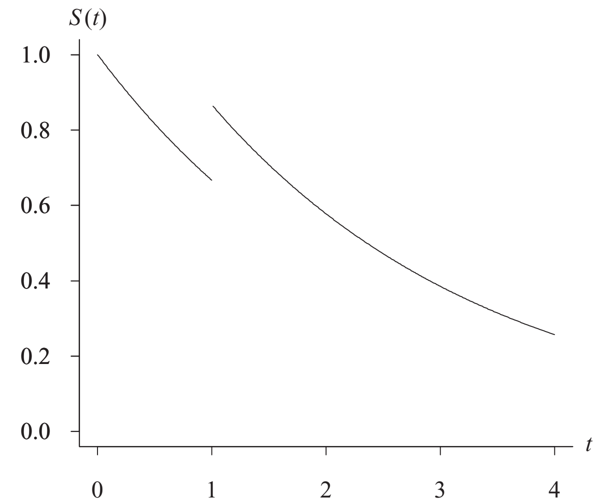

Fig. 1. Survivor function for $k = 1$ testing time $t_1 = 1$.

Example 2. As before, let $t_1 = 1$, $\lambda = -\ln(2/3)$, $n = 5$, $n_1 = 3$.

The survivor function is

$$S(t) = \begin{cases} e^{\ln(2/3)t} & \text{for } 0 < t \le 1, \\ \dfrac{2}{5} e^{\ln(2/3)t} + \dfrac{3}{5} e^{\ln(2/3)(t-1)} & \text{for } t > 1, \end{cases}$$

or

$$S(t) = \begin{cases} \left(\dfrac{2}{3}\right)^t & \text{for } 0 < t \le 1, \\ \left(\dfrac{2}{5}\right)\left(\dfrac{2}{3}\right)^t + \left(\dfrac{3}{5}\right)\left(\dfrac{2}{3}\right)^{t-1} & \text{for } t > 1, \end{cases}$$

which is plotted in Fig. 1. The upward jump at time 1 accounts for the benefit (i.e., reliability growth) associated with the testing that occurs at time $t_1 = 1$.

2.2. *Two sampling times*

When there are n_1 and n_2 items sampled at times t_1 and t_2, the survivor function is identical to the previous case up to time t_2, and is a mixture of the mixture for t values exceeding t_2:

$$S(t) = \begin{cases} e^{-\lambda t} & \text{for } 0 < t \le t_1, \\ \dfrac{n - n_1}{n} e^{-\lambda t} + \dfrac{n_1}{n} e^{-\lambda(t-t_1)} & \text{for } t_1 < t \le t_2, \\ \dfrac{n-n_2}{n}\left(\dfrac{n - n_1}{n} e^{-\lambda t} + \dfrac{n_1}{n} e^{-\lambda(t-t_1)}\right) & \text{for } t > t_2. \\ \quad + \dfrac{n_2}{n} e^{-\lambda(t-t_2)} & \end{cases}$$

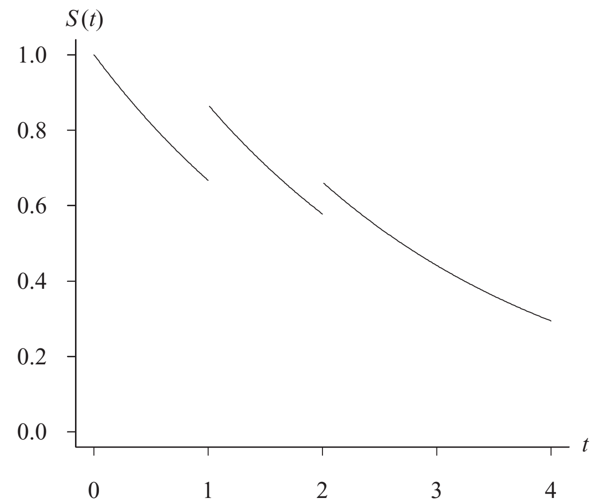

Fig. 2. Survivor function for $k = 2$ testing times $t_1 = 1$ and $t_2 = 2$.

Example 3. Let $t_1 = 1$, $t_2 = 2$, $n = 5$, $n_1 = 3$, $n_2 = 1$, $\lambda = -\ln(2/3)$. The survivor function

$$S(t) = \begin{cases} e^{\ln(2/3)t} & \text{for } 0 < t \le 1, \\ \frac{2}{5} e^{\ln(2/3)t} + \frac{3}{5} e^{\ln(2/3)(t-1)} & \text{for } 1 < t \le 2, \\ \frac{4}{5}\left(\frac{2}{5} e^{\ln(2/3)t} + \frac{3}{5} e^{\ln(2/3)(t-1)}\right) & \text{for } t > 2, \\ \quad + \frac{1}{5} e^{\ln(2/3)(t-2)} & \end{cases}$$

or

$$S(t) = \begin{cases} \left(\frac{2}{3}\right)^t & \text{for } 0 < t \le 1, \\ \frac{2}{5}\left(\frac{2}{3}\right)^t + \frac{3}{5}\left(\frac{2}{3}\right)^{t-1} & \text{for } 1 < t \le 2, \\ \frac{4}{5}\left(\left(\frac{2}{5}\right)\left(\frac{2}{3}\right)^t + \left(\frac{3}{5}\right)\left(\frac{2}{3}\right)^{t-1}\right) & \text{for } t > 2, \\ \quad + \frac{1}{5}\left(\frac{2}{3}\right)^{t-2} & \end{cases}$$

is plotted in Fig. 2. This result, and the results in Examples 1 and 2, have been verified by Monte Carlo simulation. The upward jump at $t_1 = 1$ accounts for the increase associated with the testing of $n_1 = 3$ items that occurs at time $t_1 = 1$; the smaller upward jump at $t_2 = 2$ accounts for the increase associated with the testing of just $n_2 = 1$ item that occurs at time $t_2 = 2$. The jump in $S(t)$ at $t_1 = 1$ brings the survivor function $n_1/n = 3/5$ of the vertical distance to one; the jump in $S(t)$ at $t_2 = 2$ brings the survivor function $n_2/n = 1/5$ of the vertical distance to one.

2.3. *The general case: k sampling times*

The general case (any positive integer k) is a simple extension of the $k = 2$ case, although a recursive formula will be used to define $S(t)$ for compactness. The survivor function $S(t)$ assumes a piecewise form $S_i(t)$, for $i = 1, 2, \ldots, k + 1$, due to the testing that occurs at times $t_1, t_2, \ldots, t_k$. For time values less than or equal to t_1, the first testing time, the survivor function is

$$S_1(t) = e^{-\lambda t}, \quad 0 < t \le t_1.$$

The survivor function on each subsequent time interval is defined recursively as

$$S_i(t) = \frac{n - n_{i-1}}{n} \times S_{i-1}(t) + \frac{n_{i-1}}{n} \times e^{-\lambda(t - t_{i-1})}, \quad t_{i-1} < t \le t_i,$$

for $i = 2, 3, \ldots, k + 1$, where $t_{k+1} \equiv \infty$. Even for moderate values of k, calculation of the survivor function becomes unwieldy since there are k terms in the kth piecewise segment. The following result, alluded to in the example with $k = 2$ testing times, will be used to decrease the computational time required to calculate $S(t)$.

Result 1. *For the piecewise survivor function $S(t)$ defined above:*

$$\frac{S(t_i^+) - S(t_i)}{1 - S(t_i)} = \frac{n_i}{n}, \quad i = 1, 2, \ldots, k,$$

where t_i^+ is a time value that is an infinitesimal amount larger than t_i.

The proof of a generalization of this result (Result 2) is given in the Appendix. The intuition associated with this result is that the height of the discontinuity in the survivor function at time t_i is a recovery fraction n_i/n toward one, for $i = 1, 2, \ldots, k$. In the extreme case of $n_i = n$, the survivor function resets to one, as it should, because the entire population has been sampled at time t_i.

2.4. *Varying population size*

Instances might arise when the population size does not remain constant because items might be expended, discarded or right censored. We again assume that each item has an exponential(λ) time to failure, where the failure rate $\lambda > 0$ is assumed to be a fixed constant determined from a long history of testing. Just prior to the testing times $t_1 < t_2 < \cdots < t_k$, there are $r_1, r_2, \ldots, r_k$ items "at risk" and $n_1, n_2, \ldots, n_k$ of these items are selected at random and without replacement at each sampling time from the finite population and tested, where $n_i \le r_i$, $i = 1, 2, \ldots, k$. The testing and the return to the population are again instantaneous. The net effect of the testing is to perform a perfect repair on any failed items. As before, for time values less than or equal to t_1, the survivor function is

$$S_1(t) = e^{-\lambda t}, \quad 0 < t \le t_1.$$

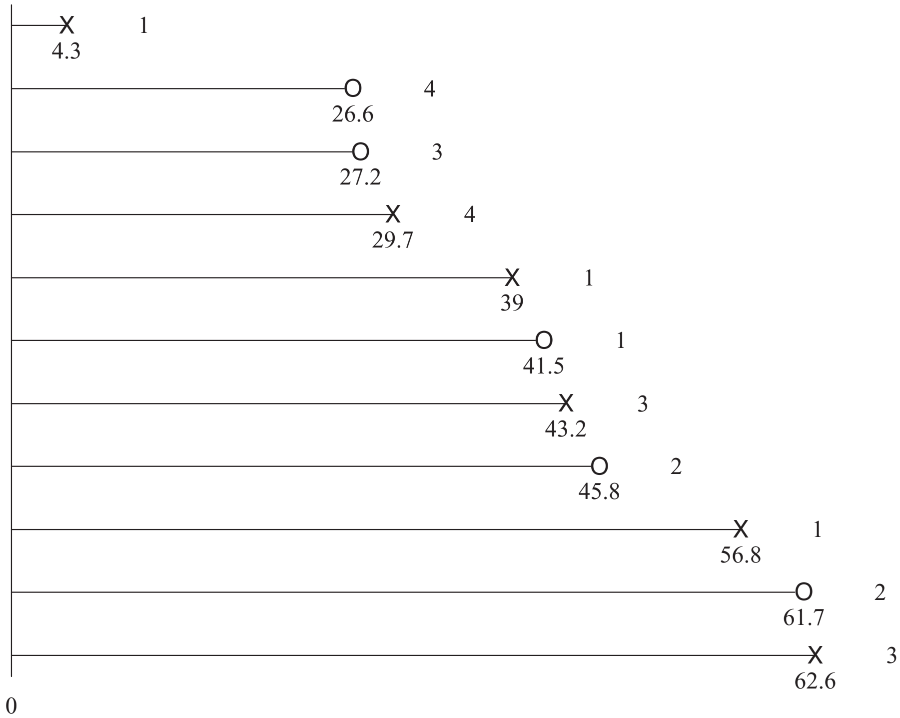

Fig. 3. Testing and right-censoring times.

The survivor function on each subsequent time interval is defined recursively as

$$S_i(t) = \frac{r_{i-1} - n_{i-1}}{r_{i-1}} \times S_{i-1}(t) + \frac{n_{i-1}}{r_{i-1}} \times e^{-\lambda(t-t_{i-1})},$$
$$t_{i-1} < t \le t_i,$$

for $i = 2, 3, \ldots, k+1$, where $t_{k+1} \equiv \infty$. An analogous result (proven in the Appendix) to the fixed population size case is given below.

Result 2. *For the piecewise survivor function $S(t)$ defined above:*

$$\frac{S(t_i^+) - S(t_i)}{1 - S(t_i)} = \frac{n_i}{r_i}, \quad i = 1, 2, \ldots, k,$$

where t_i^+ is a time value that is an infinitesimal amount larger than t_i.

Example 4. Figure 3 shows the testing and right-censoring history for a finite population of items, where time is measured in months. The × symbols correspond to testing times and the o symbols correspond to right-censoring times. The × symbols in Fig. 3 correspond to the $k = 6$ distinct testing times. There are $q = 11$ distinct testing and right-censoring times. The numbers given to the right of each line segment are the multiplicities of each observation. If these multiplicities are added, one can conclude that there are 25 unique items in the population. The goal here is to estimate the survivor function associated with this sequence of testing times and right-censoring times for a fixed failure rate λ. The right-censored values might correspond to expended items, discarded items or items currently in use. Thus, right censoring effectively corresponds to altering the population size. For each item, $t = 0$ corresponds to the purchase date.

Since each item is renewed upon testing (either by passing the test or by failing the test and being repaired), the data set can be reduced to these 25 ordered values:

4.3, 26.6*, 26.6*, 26.6*, 26.6*, 27.2*, 27.2*, 27.2*, 29.7, 29.7, 29.7, 29.7, 39.0, 41.5*, 43.2, 43.2, 43.2, 45.8*, 45.8*, 56.8, 61.7*, 61.7*, 62.6, 62.6, 62.6,

where an unmarked data value denotes a time to testing and the * superscript denotes a time to right censoring.

Let x_i be the testing or right-censoring time, $\delta_i = 0$ if the ith data value is right censored, $\delta_i = 1$ if the ith data value is a testing time, and m_i be the multiplicity of the number of distinct (x_i, δ_i) pairs. By convention, the primary sorting criterion is in ascending values of x_i (chronological) with δ_i used as a secondary sorting criterion (censored observations placed first for tied time values). This convention is necessary for the algorithm (which follows) to process the data properly. In the case of a tied censoring time and testing time, the censored items are included in the number at risk just prior to the testing time. Using this notation, the number at risk at time 0 is $\sum_{i=1}^{q} m_i$ because each testing time corresponds to a renewal. The (x_i, δ_i, m_i) triplets for the data set in this example are given in Table 1, for $i = 1, 2, \ldots, q$.

The algorithm given below is used to plot the survivor function $S(t)$ associated with a data set similar to the

Table 1. Test data ($k = 6$, $q = 11$)

i	x_i	δ_i	m_i
1	4.3	1	1
2	26.6	0	4
3	27.2	0	3
4	29.7	1	4
5	39.0	1	1
6	41.5	0	1
7	43.2	1	3
8	45.8	0	2
9	56.8	1	1
10	61.7	0	2
11	62.6	1	3

one given in the previous example. Indentation is used to denote nesting. The algorithm assumes the existence of two generic high-level plotting functions: `Plot`, for plotting a curve, and `PlotPoint`, for plotting a point. A point is plotted rather than the conventional hash mark associated with the Kaplan–Meier product–limit estimate (Kaplan and Meier, 1958) so as not to obscure the discontinuities in $S(t)$. By convention, the survivor function is not plotted after the last observed failure time. The assumption that $m_0 = 0$ is made in order to make the first pass through the loop correctly. The algorithm relies on Result 2 for the adjustment of the survivor function associated with a testing time.

Input: The failure rate λ, the number of data triplets q, and the (x_i, δ_i, m_i) triplets, for $i = 1, 2, \ldots, q$.

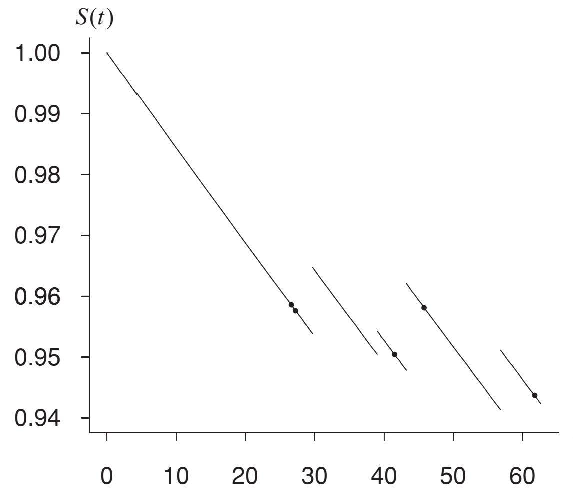

Fig. 4. Survivor function for $k = 6$ distinct testing times.

The survivor function consists of $k = 6$ piecewise segments. Although the survivor function is cut off after time 62.6, the survivor function would reset to one due to the fact that all three of the surviving items were tested at time 62.6. Survivor functions of this type will characteristically have smaller jumps in $S(t)$ for smaller values of t (e.g., $t = 4.3$ in Fig. 4) since there are more items at risk.

The data set used in the previous example can be approached from a second perspective. This new perspective allows us to consider the estimation of the finite population reliability over time.

Algorithm:

$m_0 \leftarrow 0$	initialize m_0 for first pass through loop
$t_0 \leftarrow 0$	initialize time
$S \leftarrow 1$	initialize $S(0)$
$r \leftarrow \sum_{i=1}^{q} m_i$	initialize the number at risk at time 0
for i from 1 to q	loop through data triplets
`Plot`$(Se^{-\lambda(t-t_0)},\ t_0 < t \le x_i)$	plot survivor function segment
$S \leftarrow Se^{-\lambda(x_i-t_0)}$	update $S(t)$
if $(\delta_i = 0)$	if right censored
`PlotPoint`(x_i, S)	plot a point at censoring time
else	else test time $(\delta_i = 1)$
if $(x_i = x_{i-1})$	simultaneous testing and right censoring
$S \leftarrow S + \dfrac{m_i}{r + m_{i-1}}(1 - S)$	update $S(t)$
else	distinct testing time
$S \leftarrow S + \dfrac{m_i}{r}(1 - S)$	update $S(t)$
$t_0 \leftarrow x_i$	update time
$r \leftarrow r - m_i$	decrement the number at risk

Example 5. When this algorithm is applied to the sample data from Table 1 with $\lambda = 0.0016$, the result is the plot in Fig. 4 showing the survivor function $S(t)$ on $0 < t \le 62.6$.

Example 6. Figure 5 shows the testing and right-censoring history for a finite population of 12 items. There are 25 line

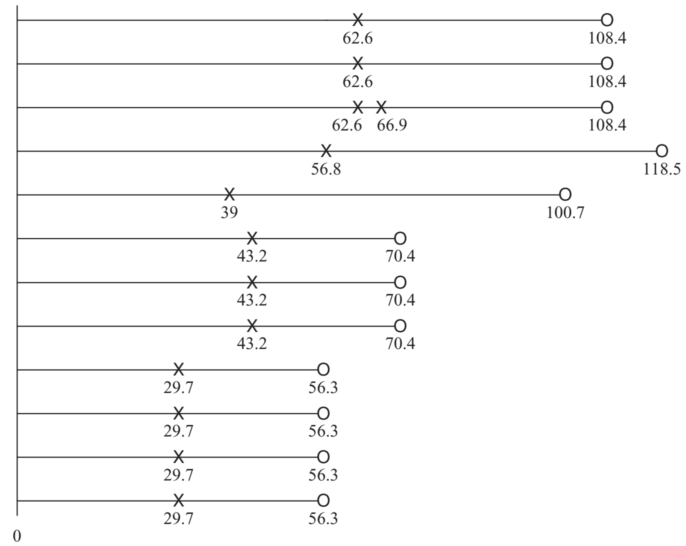

Fig. 5. Testing history for 12 repairable items.

segments shown in Fig. 5, 12 of which correspond to a right-censoring time and 13 of which correspond to a testing time. This data set, although set in a repairable context, produces the same data as that in Table 1 and the same survivor function as in Fig. 4.

3. Case study

We conclude with an analysis of the data set of testing and right-censoring times (in months) that first prompted the development of the model presented here. A long history of data collection indicates an exponential time to failure with $\lambda = 0.0015$ for a population of items. The data set consists of $q = 1094$ records with $\sum_{i=1}^{q} m_i = 2174$ distinct events (right censorings and testings). In addition:

$$\frac{\sum_{i=1}^{q} \delta_i m_i}{\sum_{i=1}^{q} m_i} = 0.226,$$

which indicates that 22.6% of the events were associated with testings; the remainder were right censorings.

Figure 6 is a plot of two survivor functions—the lower survivor function $S(t) = e^{-\lambda t}$ associated with no testing of the items in the population and the upper survivor function computed by the algorithm associated with the reliability growth accrued with the testing. The upper survivor function cuts off at the largest observed data value, 195.8, where an increase in reliability of about 0.06 has accrued due to the testing and subsequent repair of failed items in the population. The periodic testing has resulted in a significant increase in the stockpile (i.e., the finite population) reliability. The survivor function associated with the periodic testing exhibits increasing variability for larger values

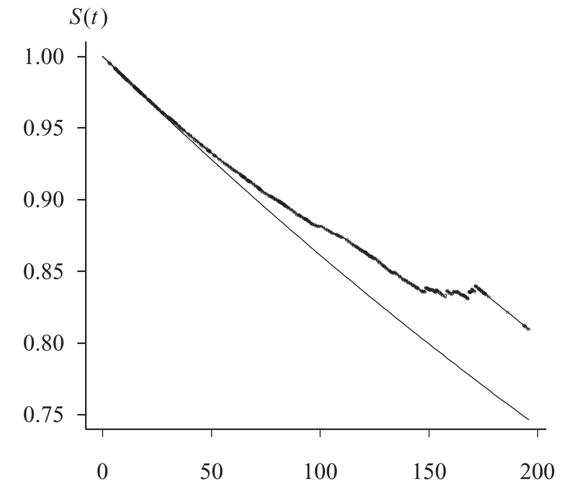

Fig. 6. Survivor function with and without testing.

of t due to the fact that the sample size is smaller in the right-hand tail of the distribution.

A second question that arose concerning this stockpile of items was to determine the frequency and fraction of items tested in order to achieve a stockpile reliability that exceeds a prescribed threshold p, where $0 < p < 1$. Assume that some fraction q of the stockpile will be sampled periodically, where $0 < q < 1$. Again assuming that λ is a known fixed constant, a stockpile of new items does not need to be tested for the first $t_0 = -(1/\lambda)\ln p$ time units (found by solving $p = e^{-\lambda t_0}$ for t_0). At time t_0, some fraction q of the population is sampled (and failed items are immediately repaired as before), which instantaneously increases the stockpile reliability to $p + q(1-p)$. The time of the next required test (to keep the stockpile reliability at or above p) can be found by solving:

$$[p + q(1-p)]e^{-\lambda(t-t_0)} = p,$$

for t. This yields a time between tests $c = t - t_0$:

$$c = -\frac{1}{\lambda}\ln\left(\frac{p}{p + q(1-p)}\right).$$

The pattern of testing a fraction q of the population every c time units continues indefinitely and results in a stockpile reliability that exceeds p. In some situations, the frequency of sampling may be dictated by the problem setting. This equation can be solved for the fraction of items sampled q yielding:

$$q = \frac{p(e^{\lambda c} - 1)}{1 - p}.$$

Both cases are illustrated numerically in the next paragraph.

Returning to the stockpile of items with exponential times to failure with a known failure rate $\lambda = 0.0015$ failures per month, we consider the calculation of c and q when the other parameter is specified for a stockpile reliability threshold $p = 0.85$. First, assume that there is the

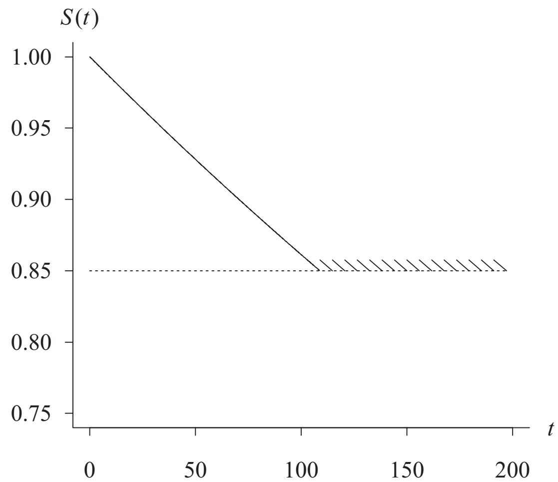

Fig. 7. Stockpile reliability at time t for $\lambda = 0.015$, $p = 0.85$ and $q = 0.05$.

ability to sample only 5% of the items at a time ($q = 0.05$); the first sampling of a group of new items should occur at

$$t_0 = -\frac{1}{\lambda} \ln p \cong 109$$

months, then every

$$c = -\frac{1}{\lambda} \ln \left(\frac{p}{p + q(1-p)} \right) \cong 5.9$$

months thereafter. The survivor function associated with these parameters is graphed on $0 < t < 200$ in Fig. 7, where the stockpile reliability always remains above $p = 0.85$. Second, consider the occasion when the sampling *time* is constrained. Assume again a failure rate $\lambda = 0.0015$ failures per month and a reliability threshold $p = 0.85$. If sampling can only occur annually ($c = 12$), then a larger proportion q of the population must be tested in order to maintain the reliability threshold:

$$q = \frac{p\left(e^{\lambda c} - 1\right)}{1 - p} \cong 0.10.$$

Hence, 10% of the population must be tested annually to maintain a long-term stockpile reliability that exceeds 0.85. Figure 8 gives a graph showing level surfaces of q and c for several values of p associated with a failure rate of $\lambda = 0.0015$ failures per month. The points considered earlier in this paragraph are plotted on the $p = 0.85$ curve. The curves match the intuition associated with the process; in order to achieve a high reliability threshold, one must sample a large fraction of the population frequently.

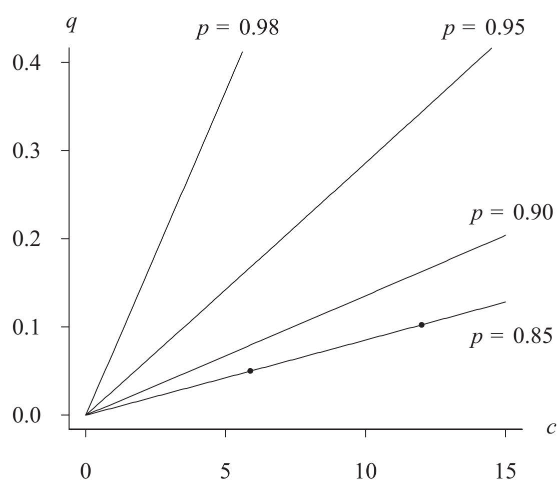

Fig. 8. Fraction tested q versus time between tests c for $\lambda = 0.015$.

4. Conclusions

The model and associated algorithm presented here yield a survivor function that reflects the reliability growth over time associated with periodic testing and repair for a finite population of statistically identical repairable items having exponential failure times with known failure rate λ. If a significant fraction of the population is tested periodically for failure, this procedure can be an effective means for keeping the stockpile reliability at a prescribed threshold.

Acknowledegment

The author gratefully acknowledges the helpful comments provided by the three referees.

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Appendix

This Appendix contains a proof of the result concerning the heights of the discontinuities in the survivor function for the case of a finite population of varying size with associated periodic testing.

Result. *Let the piecewise survivor function* $S(t)$ *be defined by*

$$S_1(t) = e^{-\lambda t}, \quad 0 < t \le t_1,$$

and

$$S_i(t) = \frac{r_{i-1} - n_{i-1}}{r_{i-1}} \times S_{i-1}(t) + \frac{n_{i-1}}{r_{i-1}} \times e^{-\lambda(t - t_{i-1})}, \quad t_{i-1} < t \le t_i,$$

for $i = 2, 3, \ldots, k+1$, where $t_{k+1} \equiv \infty$. Just prior to the testing times $t_1 < t_2 < \cdots < t_k$ when $n_1, n_2, \ldots, n_k$ items are tested, there are $r_1, r_2, \ldots, r_k$ items at risk ($n_i \le r_i$ for $i = 1, 2, \ldots, k$). Then

$$\frac{S(t_i^+) - S(t_i)}{1 - S(t_i)} = \frac{n_i}{r_i}, \quad i = 1, 2, \ldots, k,$$

where t_i^+ is a time value an infinitesimal amount larger than t_i.

Proof. Let $t_i^+ = t_i + \Delta$. The quantity of interest is

$$\frac{S(t_i^+) - S(t_i)}{1 - S(t_i)} = \lim_{\Delta \to 0} \frac{S_{i+1}(t_i + \Delta) - S_i(t_i)}{1 - S_i(t_i)}$$
$$= \lim_{\Delta \to 0} \frac{((r_i - n_i)/r_i) \cdot S_i(t_i + \Delta) + (n_i/r_i) e^{-\lambda(t_i + \Delta - t_i)} - S_i(t_i)}{1 - S_i(t_i)}$$
$$= \lim_{\Delta \to 0} \frac{(n_i/r_i)\left[e^{-\lambda\Delta} - S_i(t_i + \Delta)\right] + S_i(t_i + \Delta) - S_i(t_i)}{1 - S_i(t_i)}$$
$$= \frac{(n_i/r_i)\,[1 - S_i(t_i)] + S_i(t_i) - S_i(t_i)}{1 - S_i(t_i)}$$
$$= \frac{n_i}{r_i},$$

for $i = 1, 2, \ldots, k$, which proves the result. ■

Biography

Lawrence M. Leemis is a Professor in the Mathematics Department at the College of William & Mary. He received his B.S. and M.S. degrees in Mathematics and his Ph.D. in Industrial Engineering from Purdue University. He has also taught courses at Purdue University, The University of Oklahoma and Baylor University. He has served as an Associate Editor for *IEEE Transactions on Reliability* and *Naval Research Logistics* and Book Review Editor for the *Journal of Quality Technology*. He is a member of IIE, INFORMS and ASA.