Random Variate Generation for Monte Carlo Experiments

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Key Words—**Competing risks, Hazard function, Simulation, Thinning, Random numbers.**

Reader Aids—
Purpose: Tutorial
Special math needed for explanations: Elementary probability
Special math needed for results: Same
Results useful to: Reliability Analysts

Abstract—**We discuss methods for generating observations from specified distributions, based on a taxonomy that emphasizes analogies between methods based on the probability-density and cumulative-distribution functions and methods based on the hazard rate and cumulative hazard functions. Four categories are identified: inversion methods, linear combination methods, majorizing methods and special properties. Examples are given of each.**

1. INTRODUCTION

To estimate performance measures associated with a mathematically intractable reliability model, a reliability analyst often uses Monte Carlo simulation. This paper describes algorithms based on the probability density function and hazard rate for generating a continuous non-negative random variable. In reliability, the random variable typically represents the lifetime of an element. Multivariate and discrete versions of these algorithms can be formulated.

Our purpose is to present a taxonomy of random variate generation algorithms that highlights the analogies between those based on the probability density function and those based on the hazard function. Computational properties and trade-offs between the algorithms are not discussed since detailed information can be found in Bratley, Fox & Schrage [1], Fishman [2], Law & Kelton [5], and Schmeiser [7].

Notation

$f(t)$	pdf
$F(t)$	cumulative distribution function (Cdf)
$h(t)$	hazard rate
$H(t)$	cumulative hazard function (Chf)
$U(a, b)$	a random variable uniformly distributed on (a, b)
$\sim$	implies "is distributed as"
s-	implies "statistical(ly)"

Other, standard notation is given in "Information for Readers & Authors" at rear of each issue.

2. DISCUSSION

The term *random variate* denotes an observation of a random variable generated by transforming statistically independent $U(0, 1)$ observations for use in computer simulation experiments. Kennedy & Gentle [4] discuss $U(0,1)$ generation techniques, and the difficulties encountered when the statistical independence or uniformity assumptions are not satisfied, although they are assumed to be satisfied in this paper.

Often, random variates can be generated from a given distribution by more than one method. The choice of method should consider:

- computer round-off error
- algorithm set-up time
- marginal time to generate each random variate
- memory requirements
- number of lines of code
- portability.

The methods can be used together, eg, one method can be embedded within other methods.

Figure 1 shows the parallels between techniques using the pdf or Cdf and those using the hazard or Chf. There are four methods for variate generation: inversion, linear combination, majorizing, and special properties. Special properties is a category that exploits probabilistic relationships among random variables, and hence is not based on pdf or hazard functions.

2.1 *Inversion Methods*

The *inverse-Cdf* method generates a random variate t corresponding to the fractile u of the distribution, where $u \sim U(0, 1)$. If the Cdf has a closed form inverse, as for the Weibull distribution, then this method usually requires only one line of computer code. If the Cdf does not have a closed form inverse, as for the *s*-normal, gamma, and beta distributions, the technique can still be used by numerically integrating the pdf.

Griffith [3] shows that a random variate can be generated via the *inverse-Chf* using a unit exponential random variate as the argument. If E is an exponential random variable with mean 1, and $u \sim U(0, 1)$, then $F_E^{-1}(u) = -\ln(1 - u)$ generates E. Therefore, if H^{-1} is the inverse Chf for the random variable T, then $H^{-1}(-\ln(1 - u))$ has the desired distribution.

The inverse-Cdf and inverse-Chf methods are equivalent in the sense that two identical streams of $U(0, 1)$

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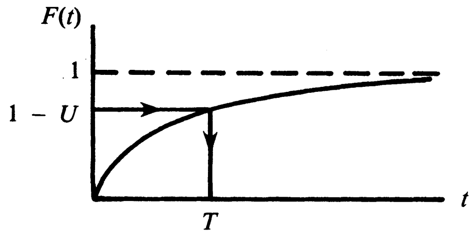
INVERSE-cdf

Preliminary

Invert the cdf $F(t)$.

Algorithm

1. Generate $U \sim U(0, 1)$
2. $T \leftarrow F^{-1}(1 - U)$



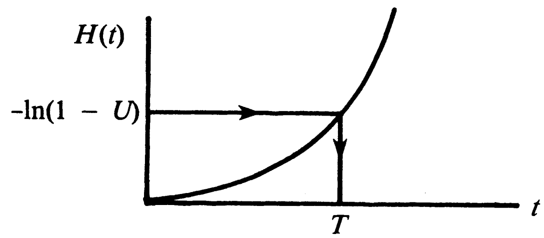
INVERSE-chf

Preliminary

Invert the chf $H(t)$.

Algorithm

1. Generate $U \sim U(0, 1)$
2. $T \leftarrow H^{-1}(-\ln(1 - U))$



COMPOSITION

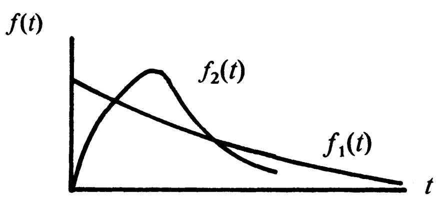
Preliminary

Express the density as

$$f(t) = \sum_{i=1}^{n} p_i f_i(t) \text{ where } \sum_{i=1}^{n} p_i = 1.$$

Algorithm

1. Choose density i with probability p_i.
2. Generate T from density i.



COMPETING RISKS

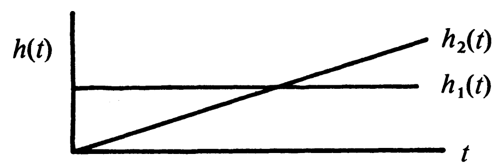
Preliminary

Express the hazard function as

$$h(t) = \sum_{i=1}^{n} h_i(t)$$

Algorithm

1. Generate $T_1, T_2, \ldots, T_n$ from $h_1(t), h_2(t), \ldots, h_n(t)$
2. $T \leftarrow \min(T_1, T_2, \ldots, T_n)$



ACCEPTANCE/REJECTION

Preliminary

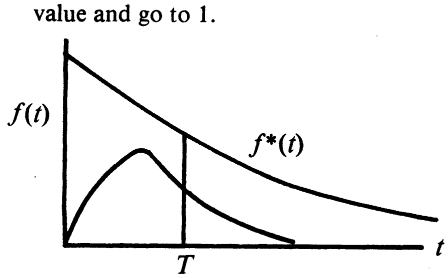
Find a majorizing function $f^*(t)$ such that $f^*(t) \geq f(t)$ for all $t > 0$.

$$\text{Let } g(t) = \frac{f^*(t)}{\int_0^\infty f^*(\tau)d\tau}$$

(Note: $g(t)$ is a density.)

Algorithm

1. Generate T from $g(t)$
2. Generate $S \sim U(0, f^*(T))$
3. If $S < f(T)$ *accept* generated T value else *reject* T value and go to 1.



THINNING

Preliminary

Find a majorizing hazard function $h^*(t)$ such that $h^*(t) \geq h(t)$ for all $t > 0$.

Algorithm

1. $T \leftarrow 0$
2. Generate Y from $h^*(y)$ given $Y > T$; $T \leftarrow T + Y$
3. Generate $S \sim U(0, h^*(T))$
4. If $S < h(T)$ *accept* generated T value else *reject* T value and go to 2.

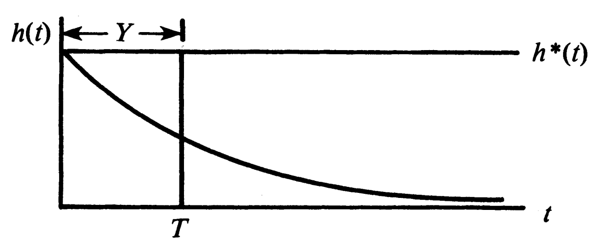


Fig. 1 Random Variate Generation Algorithms

observations produce identical random variates by the two methods. Inverse-Cdf and inverse-Chf methods require only one $U(0, 1)$ value to generate a random variate t. This property of these methods offers two advantages: order statistics are easily generated (eg, for k-out-of-n systems) and variance reduction techniques based on correlation can be employed. These advantages are discussed in Schmeiser [8].

2.2 *Linear Combination Methods*

Composition can be used when the pdf can be written as a convex combination of n other pdf's. The algorithm given in figure 1 is for discrete mixtures. The general form for composition (continuous or discrete mixtures) is:

$$f(t) = \int_{\text{all } \theta} f(t \mid \theta) dP(\theta)$$

where θ represents the parameter(s) of the distribution and $P(\theta)$ is the Cdf for θ. In this case, generate a random variate from Cdf $P(\theta)$, then generate T from $f(t \mid \theta)$.

Competing risks can apply when the hazard function can be written as a sum of n hazard functions (called risks). The lifetime T can be interpreted as failure due to the first risk to cause failure (as in a series system). Hence $T = \min\{T_1, T_2, \ldots, T_n\}$, where $T_1, T_2, \ldots, T_n$ are random variables from the n hazard functions, has the appropriate distribution. Schmeiser [9] discusses generation of T when $T_1, T_2, \ldots, T_n$ are not identically distributed.

2.3 *Majorizing Methods*

The *acceptance/rejection* technique requires finding a majorizing function $f^*(t)$ which bounds the pdf $f(t)$ from above. The majorizing function must integrate to a finite value so that it can be scaled to be a pdf, $g(t)$. Values are generated from $g(t)$, then accepted or rejected so that the accepted random variates will have pdf $f(t)$.

As suggested in Shanthikumar [10], Lewis & Shedler's [6] *thinning* algorithm, which is used to generate nonhomogeneous Poisson processes, can be modified to generate single random variates from a hazard function by restarting the algorithm at zero for each variate. As in acceptance/rejection, a majorizing function $h^*(t)$ must be found that bounds h(t) from above. Variates are generated from $h^*(t)$, by one of the previous methods, then accepted with probability $h(t)/h^*(t)$. Interestingly, when viewed in the pdf-Cdf domain, thinning is a form of composition.

2.4 *Special Properties*

Special-properties, which uses relationships between random variables to generate random variates, is related no more closely to either pdf or hazard functions. Examples include expressing:

- lognormal random variables as the exponentials of s-normals
- Erlang random variables as sums of exponentials
- gamma random variables as the product of beta and gamma random variables.

3. EXAMPLES

In each of the six examples, α and β are positive parameters. Diagrams corresponding to the examples are given in figure 1. FORTRAN code is provided with each example; T represents the generated variate, RAND (ISEED) is a function that returns a $U(0, 1)$ variate, and ISEED is an integer value which is updated by the random number generator.

Example 1 (Inverse-Cdf)

The Weibull distribution has closed form Cdf:

$$F(t) = 1 - e^{-\alpha t^\beta}$$

Set $F(t) = u$ and solve for u:

$$F^{-1}(u) = \left(-\frac{1}{\alpha} \ln(1 - u)\right)^{1/\beta}$$

Variates can be generated by one line of FORTRAN code:

```
T = (-ALOG(1.0-RAND(ISEED))/ALPHA)**(1.0/BETA)
```

Example 2 (Inverse-Chf)

Let the hazard function be $h(t) = \beta t$.

The Chf is:

$$H(t) = \beta t^2/2$$

Set $H(t) = x$ and solve:

$$H^{-1}(x) = \sqrt{2x/\beta}$$

Variates can be generated using a unit exponential variate as an argument in the inverse Chf:

```
T = SQRT(-2.0*ALOG(1.0-RAND(ISEED))/BETA)
```

Example 3 (Composition)

In a manufacturing process, 30% of the components have exponential lifetimes with a mean of 0.5, and 70% of the components have Weibull lifetimes with $\alpha = 3$ and $\beta = 4$. If a component is selected at random, the mixture model:

$$f(t) = (0.3)(2)\exp(-2t) + (0.7)(12)t^3\exp(-3t^4)$$

is appropriate. Use inversion to generate each individual population lifetime. The code to generate variates is:

```
E = -ALOG (1.0-RAND(ISEED))
IF (RAND(ISEED).LT.0.3) THEN
        T = E/2.0
    ELSE
        T = (E/3.0)**0.25
    ENDIF
```

Example 4 (Competing Risks)

Let the hazard function be the sum:

$$h(t) = \alpha + \beta t.$$

Then the α and β terms can be interpreted as two competing risks. The first risk, $h_1(t) = \alpha$ corresponds to constant rate failures. The second risk, $h_2(t) = \beta t$ corresponds to failures which are more likely as time passes (such as wearout). The code to generate variates is:

```
T1 = -ALOG(1.0-RAND(ISEED))/ALPHA
T2 = SQRT (-2.0*ALOG(1.0-RAND(ISEED))/BETA)
T  = AMIN (T1, T2)
```

Example 5 (Acceptance/Rejection)

The Weibull distribution with unity scale parameter and a shape parameter of 2 has pdf:

$$f(t) = 2t\exp(-t^2)$$

that is majorized by:

$$f^*(t) = 3\exp(-t)$$

so that —

$$g(t) = \frac{3e^{-t}}{\int_0^\infty 3e^{-t}\,dt} = e^{-t}$$

is a pdf. Variates can be generated by:

```
10 T = -ALOG(RAND(ISEED))
   S = RAND(ISEED)*3.0*EXP(-T)
   F = (T + T)*EXP(-T*T)
   IF(S.GT.F) GO TO 10
```

Example 6 (Thinning)

The hazard rate

$$h(t) = \alpha/(1 + \alpha t)$$

is monotone decreasing from α, so it can be majorized by $h^*(t) = \alpha$. Since the constant hazard rate corresponds to an exponential distribution, variates are easily generated from $h^*(t)$ by the inverse-Cdf method. Thus the thinning algorithm code is:

```
   T = 0
10 Y = -ALOG(1.0-RAND(ISEED))/ALPHA
   T = T + Y
   S = ALPHA*RAND(ISEED)
   H = ALPHA/(1.0 + ALPHA*T)
   IF (S.GT.H) GO TO 10
```

As suggested in Shanthikumar [10], this algorithm can be made more efficient by decreasing the hazard rate if T is rejected as indicated in figure 2. The new code is:

```
   T = 0
   H = ALPHA
10 Y = -ALOG(1.0-RAND(ISEED))/H
   T = T + Y
   S = H*RAND(ISEED)
   H = ALPHA/(1.0 + ALPHA*T)
   IF (S.GT.H) GO TO 10
```

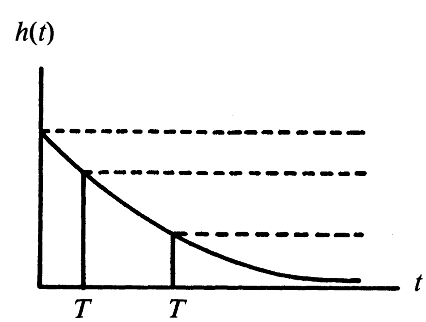


Fig. 2. Thinning for a DFR distribution.

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