THE CROSSING NUMBER METHOD

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In 1997 [*Combinatorics, Probability, and Computing* **6**(3)(1997), 353–358] I observed an unexpected connection between incidences of points and lines, and the crossing number of graphs. Using this connection, I gave short proofs, using a lemma from the theory of crossing number, to the following results. (1) the Szemerédi–Trotter theorem: a tight upper bound of $O(n+m+(nm)^{2/3})$ on the number of incidences among n points and m lines in the plane. (2) the Szemerédi–Trotter theorem: an upper bound of $O(n^{4/3})$ on the number of unit distances among n points in the plane. (3) the Chung–Szemerédi–Trotter theorem: a lower bound of $\Omega(n^{.8})$ on the number of distinct distances among n points in the plane.

Since 1997 the crossing number method has been applied, mostly by Dey, Pach, and Sharir, to a number of other problems—it has been cited in 68 papers. Problem (3) has been subject to subsequent improvements, all combining the crossing number method with other arguments, reaching the current $\Omega(n^{.8636})$ by Katz and Tardos. (The conjecture of Erdős from 1945, $\Omega(n/\sqrt{\log n})$, is out of reach of the existing methods.)

I will complete the proofs with a folklore probabilistic proof to the crossing number lemma, and will show applications of the Szemerédi–Trotter theorem.

I will discuss avenues for possible generalizations of the Szemerédi–Trotter theorem. One of them is the following. Counting incidences among points and lines is equivalent to counting edges in the incidence bipartite graph, in which the vertex classes are the sets of points and lines, and edges represent incidences. Good upper bounds for the number of 3-paths (solved, in collaboration with de Caen, using the second moment method) or 6-cycles (unsolved) provide generalization of the Szemerédi–Trotter theorem.

This talk is suitable for advanced undergraduates.