Part 5 Functions and Matrices

We study functions on matrices related to the Löwner (positive semidefinite) ordering on positive semidefinite matrices that $A \geq B$ means $A - B$ is positive semi-definite. Note that every function $f : \mathbf{C} \to \mathbf{C}$ can be extended to diagonalizable matrices using the spectral decomposition

$$A = S^{-1}\text{diag}\,(a_1, \ldots, a_n)S \mapsto S^{-1}\text{diag}\,(f(a_1), \ldots, f(a_n))S.$$

More generally, if A has minimal polynomial $(z - a_1)^{r_1} \cdots (z - a_k)^{r_k}$, and f has derivatives at a neighborhoods of a_j up to order $r_j - 1$, then $f(A)$ can be defined, and has the same value as $p(A)$ for a polynomial interpolating the values and derivatives of $f(a_j)$ up to order $r_j - 1$ for $j = 1, \ldots, k$.

1 Positive and completely positive linear maps

Definition 1.1 *A function $\Phi : M_n \to M_m$ is a* **positive linear map** *if $\Phi(A) \geq \Phi(B)$ whenever $A \geq B$. It is unital if $\Phi(I) = I$. It* k**-positive** *if*

$$\Phi \otimes I_k(A_{ij})_{1 \leq i,j \leq k}) = (\Phi(A_{ij}))_{1 \leq i,j \leq k}$$

is positive whenever $(A_{ij}) \in M_k(M_n))$ is positive. If Φ is k-positive for all $k = 1, 2, \ldots,$ then Φ is **completely positive**.

Lemma 1.2 *Let $A, B \in H_n$.*

(a) *Then $A \leq B$ if and only if $X^*AX \leq X^*BX$ for any nonzero $n \times m$ matrix X.*

(b) *Suppose $0 \leq A, B$ and B is invertible. Then $A \leq B$ is and only if $\|A^{1/2}B^{-1/2}\| \leq 1$.*

Lemma 1.3 *A matrix $C = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \in H_n$ is positive semi-definite if and only if $B - X^*A^{-1}X$ is positive semi-definite.*

Lemma 1.4 *Suppose Φ is a positive linear map. If $A, B, C \in H_n$ are such that $\begin{pmatrix} A & C \\ C & B \end{pmatrix} \geq 0$, then $\begin{pmatrix} \Phi(A) & \Phi(C) \\ \Phi(C) & \Phi(B) \end{pmatrix} \geq 0$.*

Corollary 1.5 *Suppose Φ is a positive linear map. If $A \geq 0$, then*

$$\Phi(A^2) \geq \phi(A)^2 \quad \textit{and} \quad \Phi(A^{-1}) \geq \Phi(A)^{-1}.$$

Remark 1.6 *For a fixed $n \times m$ matrix X such that $X^*X = I_m$, the map defined by $\Phi(A) = X^*AX$ is unital and completely positive.*

Suppose $P_1, \dots, P_k \in M_n$ are orthogonal projections such that $P_1 + \cdots + P_k = I$. The map $A \mapsto P_1AP_1 + \cdots + P_kAP_k$ is unital and completely positive.

Theorem 1.7 *Let $\Phi : M_n \to M_m$. The following are equivalent.*

(a) *Φ is completely positive.*

(b) *$(\Phi(E_{ij}))_{1 \le i,j \le n}$ is positive semidefinite.*

(c) *There are $n \times m$ matrices $X_1, \dots, X_k$ such that*

$$\Phi(A) = \sum_{j=1}^{k} X_j^* A X_j.$$

2 Operator monotone and operator convex functions

Definition 2.1 *Let $f : \mathbf{R} \to \mathbf{R}$.*

(a) *The function f is* **matrix monotone of order n** *if for all $A, B \in H_n$ and all $t \in [0, 1]$,*

$$f(A) \le f(B) \quad \textit{whenever} \quad A \le B.$$

If this is true for all orders, then f is **operator monotone**.

(b) *The function f is* **matrix convex of order n** *if for all $A, B \in H_n$ and all $t \in [0, 1]$,*

$$f((1-t)A + tB) \le (1-t)f(A) + tf(B).$$

If this is true for all orders, then f is **operator monotone**.

If the condition on f holds for $t = 1/2$, then f is **mid-point matrix/operator convex**. *For continuous functions, mid-point convex implies matrix/operator convex.*

(c) *The function f is* **matrix concave of order n** *if $-f$ is matrix convex of order n.*

Example 2.2 *Here are some basic examples.*

1. *The function $f(t) = c + dt$ is operator monotone and operator convex if $d \ge 0$ and $c \in \mathbf{R}$. It is operator convex for all $c, d \in \mathbf{R}$.*

2. *The function $f(t) = t^2$ is operator convex, but not operator monotone.*

3. *The function $f(t) = t^3$ is not operator convex, and not operator monotone.*

4. *The function $f(t) = |t|$ is not operator convex.*

Theorem 2.3 (a) $f(t) = t^{-1}$ *is operator monotone on* $(0, \infty)$.

(b) *If* $r \in [0, 1]$, *then* $f(t) = t^r$ *is operator monotone on* $[0, \infty)$.

Theorem 2.4 *If* f *is operator monotone on* $[0, \infty)$, *then there are* $a, b \in \mathbf{R}$ *with* $b \geq 0$ *and a positive measure* μ *on* $[0, \infty)$ *such that*

$$f(t) = a + bt + \int_0^\infty \frac{st}{s+t} d\mu(s).$$

If f *is operator convex on* $[0, \infty)$, *then there are* $a, b, c \in \mathbf{R}$ *with* $c \geq 0$ *and a positive measure* μ *on* $[0, \infty)$ *such that*

$$f(t) = a + bt + ct^2 + \int_0^\infty \frac{st^2}{s+t} d\mu(s).$$

Theorem 2.5 *Let* $A, B \in H_n$ *be positive semidefinite, and* $\|\cdot\|$ *be a UI norm. If* $f(t)$ *is a nonnegative operator monotone function* $f(t)$ *on* $[0, \infty)$, *then*

$$\|f(A+B)\| \leq \|f(A) + f(B)\|.$$

Furthermore, if f^{-1} *exists such that* $f(0) = 0$ *and* $f(\infty) = \infty$, *then*

$$\|f^{-1}(A+B)\| \geq \|f^{-1}(A) + f^{-1}(B)\|.$$

Corollary 2.6 *Let* $\|\cdot\|$ *be a UI norm on* M_n. *If* $A, B \in H_n$ *are positive semidefinite and* $r \in (0, 1]$ *then*

$$\|(A+B)^r\| \leq \|A^r + B^r\| \qquad \|(A+B)^{1/r}\| \geq \|A^{1/r} + B^{1/r}\|,$$

and

$$\|\log(A+B+I)\| \leq \|\log(A+I) + \log(B+I)\| \quad \text{and} \quad \|e^A + e^B\| \leq \|e^{A+B} + I\|.$$

It is known that a function $g(t)$ on $[0, \infty)$ with $g(0) = 0$ is operator convex if and only if $g(t)/t$ is operator monotone on $(0, \infty)$. We have the following corollary.

Corollary 2.7 *Let* $g(t)$ *be a nonnegative operator convex function on* $[0, \infty)$ *with* $g(0) = 0$. *If* $\|\cdot\|$ *is a UI norm on* M_n, *then*

$$\|g(A+B)\| \geq \|g(A) + g(B)\|$$

for any positive semidefinite $A, B \in H_n$.

Theorem 2.8 *Let* $f(t)$ *be a nonnegative operator monotone function on* $[0, \infty)$ *and* $\|\cdot\|$ *be a UI norm on* M_n. *Let* $A \in M_n$.

(a) *If* $\|\mathrm{diag}\,(1, 0, \ldots, 0)\| = 1$, *then*

$$f(\|A\|) \leq f(|A|)\|.$$

(b) *If* $\|I\| = 1$, *then*

$$f(\|A\|) \geq f(|A|)\|.$$

3 Extremal representation techniques

Proposition 3.1 *Let $A \in H_n$ be positive definite, and $B \in M_n$. Then*

$$B^*A^{-1}B = \min\left\{C : \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0\right\}.$$

In particular,

$$A^{-1} = \min\left\{C : \begin{pmatrix} A & I \\ I & C \end{pmatrix} \geq 0\right\}.$$

Corollary 3.2 *If $A \geq B$ then $A^{-1} \leq B^{-1}$.*

Proposition 3.3 *Note that*

$$A^{1/2} = \max\left\{C : \begin{pmatrix} A & C \\ C & I \end{pmatrix} \geq 0\right\}.$$

Thus, $A \geq B$ implies that $A^{1/2} \geq B^{1/2}$.

Proposition 3.4 *Suppose $A = (A_{ij})_{1\leq i,j\leq 2} \in H_n$ is positive definite. Then the Schur-complement of A with respect to A_{11} given by $S(A) = A_{22} - A_{21}^* A_{11}^{-1} A_{12}$ satisfies*

$$S(A) = \max\{C : A \geq 0_k \oplus C \text{ with } C \in H_{n-k}\}$$

and

$$S(A) = \min\{[Z|I_{n-k}]A[Z|I_{n-k}]^* : Z \text{ is } (n-k)\times k\}.$$

Corollary 3.5 *Suppose f is operator monotone on $[0,\infty)$ and $f(0) \geq 0$. Then for any positive linear map from M_n to M_{n-k}, and any positive definite matrix A,*

$$f(\Phi(A)) \geq \Phi(f(A)) \geq S(f(A)) \geq f(S(A)).$$

In particular,

$$[\Phi(A^p)]^{1/p} \geq \Phi(A) \geq S(A) \geq [S(A^p)]^{1/p} \quad \text{for } p \geq 1,$$

and

$$[\Phi(A^p)]^{1/p} \leq \Phi(A) \leq S(A) \leq [S(A^p)]^{1/p} \quad \text{for } p \leq -1.$$

Corollary 3.6 *Suppose $A \in H_n$ is positive definite and $B \in M_n$. Then for any $1 \leq i_1 < \cdots < i_m \leq k$ and $1 \leq j_1 < \cdots < j_m \leq n$. If $i_m + j_m \leq m + k$, then*

$$\prod_{s=1}^{m} \lambda_{i_s+j_s-s}(S(BAB^*)) \leq \prod_{s=1}^{m} \lambda_{i_s}(S(BB^*))\lambda_{j_s}(A).$$

4 Current research

Theorem 4.1 *Suppose* $f : [0, \infty) \to [0, \infty)$ *is concave. If* $A, B \in H_n$ *are positive semidefinite, then*

$$\|f(A + B)\| \leq \|f(A) + f(B)\|$$

for any UI norm.

Problem 4.2 *Suppose* $f : [0, \infty) \to [0, \infty)$ *is concave. If* $A, B \in M_n$ *are normal. Is it true that*

$$\|f(|A + B|)\| \leq \|f(|A|) + f(|B|)\|$$

5 Exercises

1. Suppose Φ is a positive linear map. Show that

$$\|\Phi\| = \max\{\|\Phi(A)\| : s_1(A) \leq 1\} = \|\Phi(I)\|.$$

2. Show that $\|\Phi\|$ is attained at a unitary, and show that

$$\begin{pmatrix} \Phi(I) & \Phi(U) \\ \Phi(U)^* & \Phi(I) \end{pmatrix} \geq 0$$

   to conclude that $\|\Phi(U)\| \leq \|\Phi(I)\|$ for any unitary U.

3. Suppose Φ is a positive linear map such that $\Phi(I) \leq I$. Show that $\|\Phi(A)\| \leq \|A\|$ for any UI norm $\| \cdot \|$.

4. Show that the scalar function $f(x) = |x|$ on M_1 is 2-positive but not 3-positive.

5. Show that the scalar function $f(x) = |x|$ is not operator convex.