Part 2 Submatrices, Sum and Product of Matrices

In perturbation theory, one often study the change of eigenvalues and singular values of sum and product of matrices. Here we present some basic results and techniques.

1 Sum of Matrices

Theorem 1.1 (Lidskii) *Let $A, B \in H_n$ has eigenvalues $a_1 \geq \cdots \geq a_n$ and $b_1 \geq \cdots \geq b_n$, respectively. Suppose $C = A + B$ has eigenvalues $c_1 \geq \cdots \geq c_n$. For any $1 \leq i_1 < \cdots < i_m$,*

$$\sum_{s=1}^{m} b_{n-j+1} \leq \sum_{s=1}^{m} (c_{i_s} - a_{i_s}) \leq \sum_{s=1}^{m} b_s.$$

More generally, we have the following.

Theorem 1.2 (Thompson) *Let $A, B \in H_n$ have eigenvlues $a_1 \geq \cdots \geq a_n$ and $b_1 \geq \cdots \geq b_n$, respectively. Suppose $C = A + B$ has eigenvalues $c_1 \geq \cdots \geq c_n$. If $1 \leq i_1 < \cdots < i_m$ and $1 \leq j_1 < \cdots < j_m \leq n$, then*

$$\sum_{s=1}^{m} (a_{i_s} + b_{j_s}) \geq \sum_{s=1}^{m} c_{i_s + j_s - s}.$$

Theorem 1.3 *Let $A \in M_n$ have singular values $s_1 \geq \cdots \geq s_n$. Then*

$$\tilde{A} = \begin{pmatrix} 0_n & A \\ A^* & 0_n \end{pmatrix} \in H_{2n}$$

have eigenvalues $\pm s_1, \ldots, \pm s_n$.

Theorem 1.4 *Let $A, B \in M_n$ have singular values $a_1 \geq \cdots \geq a_n$ and $b_1 \geq \cdots \geq b_n$, respectively. Suppose $C = A + B$ has singular values $c_1 \geq \cdots \geq c_n$. Then for any $1 \leq i_1 < \cdots < i_m$ and $1 \leq j_1 < \cdots < j_m \leq n$,*

$$\sum_{s=1}^{m} (a_{i_s} + b_{j_s}) \geq \sum_{s=1}^{m} c_{i_s + j_s - s}.$$

2 Product of Matrices

Theorem 2.1 *Let $A, B \in H_n$ have eigenvalues $a_1 \geq \cdots \geq a_n$ and $b_1 \geq \cdots \geq b_n$.*

(a) *There exists an invertible S such that $B = S^*AS$ if and only if A and B have the same inertia.*

(b) *If there exists $S \in M_n$ with $\lambda_n(S^*S) \geq 1$ such that $B = S^*AS$, then $|a_j| \leq |b_j|$ for all $j = 1, \ldots, n$.*

Theorem 2.2 *Let $S \in M_n$ be invertible and $A \in H_n$. Then $\tilde{A} = S^*AS$ and A have the same inertia. If $1 \leq i_1 < \cdots < i_m \leq n$ are such that*

$$\prod_{j=1}^{m} \lambda_{i_j}(A) \neq 0 \quad \text{or / and} \quad \prod_{j=1}^{m} \lambda_{i_j}(\tilde{A}) \neq 0,$$

then

$$\prod_{j=1}^{m} \lambda_{n-j+1}(S^*S) \leq \prod_{j=1}^{m} [\lambda_{i_j}(\tilde{A})/\lambda_{i_j}(A)] \leq \prod_{j=1}^{m} \lambda_j(S^*S).$$

Theorem 2.3 *Let $A, B \in M_n$ have singular values $a_1 \geq \cdots \geq a_n$ and $b_1 \geq \cdots \geq b_n$, respectively. Suppose $C = AB$ has singular values $c_1 \geq \cdots \geq c_n$. Then for any $1 \leq i_1 < \cdots < i_m$ and $1 \leq j_1 < \cdots < j_m \leq n$,*

$$\prod_{s=1}^{m} (a_{i_s} b_{j_s}) \geq \prod_{s=1}^{m} c_{i_s+j_s-s}.$$

3 Submatrices

Theorem 3.1 *Let $A \in H_n$ and $U_1, \ldots, U_k \in M_n$ be unitary. Then*

$$\frac{1}{k} \lambda \left(\sum_{j=1}^{k} U_j^* A U_j \right) \prec \lambda(A).$$

Corollary 3.2 *If $A = (A_{ij})_{1 \leq i,j \leq k} \in H_n$, then*

$$\lambda(A_{11} \oplus \cdots \oplus A_{kk}) \prec \lambda(A).$$

Lemma 3.3 *Let A and B be $m \times n$ matrices. Then AB and BA have the same nonzero eigenvalues.*

Theorem 3.4 *Suppose* $C = (C_{ij})_{1 \le i,j \le 2} \in H_n$ *has eigenvalues* $c_1 \ge \cdots \ge c_n$, $C_{11} \in H_k$ *has eigenvalues* $a_1 \ge \cdots \ge a_k$, *and* $C_{22} \in H_{n-k}$ *has eigenvalues* $b_1 \ge \cdots \ge b_{n-k}$. *Set* $a_i = c_n$ *for* $i \in \{k+1, \dots, n\}$ *and* $b_j = c_n$ *for* $j \in \{n-k+1, \dots, n\}$. *Then for any* $1 \le i_1 < \cdots < i_m \le n$ *and* $1 \le j_1 < \cdots < j_m \le n$,

$$\sum_{s=1}^{m} [(a_{i_s} - c_n) + (b_{j_s} - c_n)] \ge \sum_{s=1}^{m} (c_{i_s + j_s - 1} - c_n).$$

Remark 3.5 *We can obtain inequalities relating the singular values of* C_{12} *and the eigenvalues of* C *using the fact that*

$$2 \begin{pmatrix} 0_k & C_{12} \\ C_{12}^* & 0_{n-k} \end{pmatrix} = C - (I_k \oplus -I_{n-k}) C (I_k \oplus -I_{n-k}).$$

4 Cartesian decomposition

Theorem 4.1 *Let* $A, B \in H_n$ *have singular values* $a_1 \ge \cdots \ge a_n$ *and* $b_1 \ge \cdots \ge b_n$. *If* $A + iB$ *has eigenvalues* $z_1, \dots, z_n$, *then*

$$\mathrm{Re}(z_1^2, \dots, z_n^2) \prec (a_1^2 - b_n^2, \dots, a_n^2 - b_1^2).$$

Theorem 4.2 *Let* $A, B \in H_n$ *have singular values* $a_1 \ge \cdots \ge a_n$ *and* $b_1 \ge \cdots \ge b_n$. *Suppose* $A + iB$ *has singular values* $s_1, \dots, s_n$. *Then*

$$(a_1^2 + b_n^2, \dots, a_n^2 + b_1^2) \prec (s_1^2, \dots, s_n^2) \quad \text{and} \quad (s_1^2 + s_n^2, \dots, s_n^2 + s_1^2) \prec 2(a_1^2 + b_1^2, \dots, a_n^2 + b_n^2).$$

Proposition 4.3 *Let* $A \in M_n$ *have singular values* $s_1 \ge \cdots \ge s_n$. *For* $k \in \{1, \dots, n\}$,

$$\sum_{j=1}^{k} s_j = \min \left\{ \sum_{j=1}^{n} s_j(B) + k s_1(C) : B, C \in M_n,\ A = B + C \right\}.$$

Theorem 4.4 *Using the notation in Theorem 4.2, we have*

$$\frac{1}{\sqrt{2}} (s_1, \dots, s_n) \prec_w (|a_1 + ib_1|, \dots, |a_n + ib_n|) \prec_w \sqrt{2} (s_1, \dots, s_n).$$

Theorem 4.5 *Suppose* $A, B \in H_n$ *have singular values* $a_1 \ge \cdots \ge a_n$ *and* $b_1 \ge \cdots \ge b_n$. *Then*

$$|\det(A + iB)| \le \prod_{j=1}^{n} |a_j + ib_{n-j+1}|.$$

If A *and* B *are positive definite, then*

$$\prod_{j=k}^{n} s_j(A + iB) \ge \prod_{j=k}^{n} |(a_j + ib_j)|.$$

5 Tensor products

Definition 5.1 *Let $A = (a_{ij})$ and $B = (b_{ij})$ be matrices. The* **tensor product** *of A and B is the matrix $A \otimes B = (a_{ij}B)$. If A and B are of the same size, then the* **Schur product** *of A and B is the matrix $A \circ B = (a_{ij}b_{ij})$, which is a submatrix of $A \otimes B$.*

Theorem 5.2 *Let $A \in M_n$ and $B \in M_m$ have eigenvalues (respectively, singular values) $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$. Then $A \otimes B$ have eigenvalues (respectively, singular values) $a_i b_j$ with $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$.*

Corollary 5.3 *If $A, B \in H_n$ are positive definite, then*

$$\lambda_n(A \circ B) \geq \lambda_n(A)\lambda_n(B).$$

Theorem 5.4 *Suppose $A \in M_r$ and $B \in M_s$ have eigenvalues $a_1, \ldots, a_r$ and $b_1, \ldots, b_s$. Then $A \otimes I_s + I_r \otimes B$ have eigenvalues $a_i + b_j$ with $(i, j) \in \{1, \ldots, r\} \times \{1, \ldots, s\}$.*

Corollary 5.5 *Suppose $p(z)$ and $q(z)$ are the integral polynomials with the algebraic numbers a and b as zeros. Let $A \in M_r$ and $B \in M_s$ be the companion matrices for $p(z)$ and $q(z)$. Then ab is a zero of $A \otimes B$, and $a + b$ is a zero of $A \otimes I_s + I_r \otimes B$.*

6 Additional results and open problems

The necessary and sufficient condition has been determined for the existence of Hermitian A, B and $C = A + B$ with eigenvalues $a_1 \geq \cdots \geq a_n$, $b_1 \geq \cdots \geq b_n$, and $c_1 \geq \cdots \geq c_n$, respectively. The condition is described in term of the equality $\sum_{j=1}^{n}(a_j + b_j) = \sum_{j=1}^{n} c_j$ and inequalities of the form

$$\sum_{r \in R} a_r + \sum_{s \in S} b_s \geq \sum_{t \in T} c_t$$

for a collection of subsets R, S, T of $\{1, \ldots, n\}$.

There are similar results on

(a) the relations of the singular values of sum and product of matrices.

(b) the relations of the eigenvalues and singular values of submatrices and the entire matrix.

(c) the relations of singular values of A_{12} and the eigenvalues of $A = (A_{ij})_{1 \leq i,j \leq 2} \in H_n$.

(d) the relations of the diagonal entries, eigenvalues, and singular values of sum and product of matrices.

There are additional results concerning the determinant, the rank, the eigenvalues, the inertia and the norms of sum of matrices from unitary orbits.

There are many problems under **current research**.

1. Determine the complete set of eigenvalues (respectively, singular values, inertia values, ranks and norm values) of $U^*AU + V^*BV$ for unitary $U, V \in M_n$.

2. More generally, one may consider the above problems for square matrices $A \in M_n$ and $B \in M_m$, and use partial isometries U and V such that $U^*U = I_k$ and $V^*V = I_k$. If A and B are adjacency matrices of two graphs, then the above problem is related to finding similar subgraphs in the two given graphs.

3. Let $A = (a_{ij}) \in M_n$ be real symmetric. Determine orthogonal matrices $Q_1, \ldots, Q_n \in M_n$ such that
$$\operatorname{diag}(a_{11}, \ldots, a_{nn}) = n^{-1} \sum_{j=1}^{n} Q_j^t A Q_j.$$
Note that
$$A = 2^{1-n} \left(\sum_{j=1}^{2^{n-1}} D_j A D_j \right),$$
where $D_1, \ldots, D_{2^{n-1}}$ are all diagonal orthogonal matrices with $(1,1)$ entry equal to 1.

4. Determine the condition on C_{12}, C_{13}, C_{23} for the existence of $C = (C_{ij})_{1 \le i,j \le 3} \in H_n$ with prescribed eigenvalues $c_1 \ge \cdots \ge c_n$.

5. **Riemann hypothesis** can be formulated as a problem of estimating the determinant. Let $D_n = (d_{ij}) \in M_n$ be the divisor matrix defined by $d_{ij} = 1$ if j is a multiple of i, and $d_{ij} = 0$ otherwise. Let $L_n \in M_n$ be the matrix with 1 at the $(i, 1)$ entry for $i = 2, \ldots, n$, and all other entries equal to 0. If $A_n = D_n + L_n$, then
$$\det(A_n) = \sum_{j=1} \mu(j)$$
is the Mertens' function, i.e., $\mu(j)$ is the Möbius function. Hence, Riemann hypothesis is true if and only if
$$|\det(A_n)| = O(n^{1/2+\varepsilon}) \quad \text{for every } \varepsilon > 0.$$

7 Exercises

1. Fill in the many missing details in our discussion.

2. Let $A, B \in M_n(\mathbf{C})$ have singular values $a_1 \geq \cdots \geq a_n$ and $b_1 \geq \cdots \geq b_n$. Show that

$$\prod_{j=1}^{n}(a_j + b_{n-j+1}) \geq |\det(A+B)| \geq \begin{cases} 0 & \text{if } [a_n, a_1] \cap [b_n, b_1] \neq \emptyset, \\ \prod_{j=1}^{n} |a_j - b_{n-j+1}| & \textit{otherwise.} \end{cases}$$

3. Let $A = \begin{pmatrix} R & 0 \\ S & T \end{pmatrix}$ has singular values $s_1 \geq \cdots \geq s_{2n} > 0$, where $R, S, T \in M_n$. Show that

$$s(T^{-1}SR^{-1}) \prec_w \left(s_{2n}^{-1} - s_1^{-1}, \ldots, s_{n+1}^{-1} - s_n^{-1}\right),$$

$$s(SR^{-1}) \prec_w \frac{1}{2}\left(\frac{s_1}{s_{2n}} - \frac{s_{2n}}{s_1}, \ldots, \frac{s_n}{s_{n+1}} - \frac{s_{n+1}}{s_n}\right),$$

and

$$s(T^{-1}S) \prec_w \frac{1}{2}\left(\frac{s_1}{s_{2n}} - \frac{s_{2n}}{s_1}, \ldots, \frac{s_n}{s_{n+1}} - \frac{s_{n+1}}{s_n}\right).$$

4. Let $A_{11}, A_{12}, A_{21}, A_{22} \in M_n$ and $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in H_{2n}$ have eigenvalues $a_1 \geq \cdots \geq a_{2n} > 0$. Show that

$$s(A_{22}^{-1/2}A_{21}) \prec_w \left(\sqrt{a_1} - \sqrt{a_{2n}}, \ldots, \sqrt{a_n} - \sqrt{a_{n+1}}\right),$$

and

$$s(A_{21}A_{11}^{-1}) \prec_w \left(\sqrt{\frac{a_1}{a_{2n}}} - \sqrt{\frac{a_{2n}}{a_1}}, \ldots, \sqrt{\frac{a_n}{a_{n+1}}} - \sqrt{\frac{a_{n+1}}{a_n}}\right).$$