REPRESENTATIONS OF LOCALLY COMPACT TOTALLY DISCONNECTED GROUPS

MATH 519

1. Smooth and admissible representations

Let G be a locally compact totally disconnected group. Recall that every neighborhood of 1 in G contains a compact open subgroup. If G is assumed to be compact, then every neighborhood of 1 in G contains a compact open normal subgroup. On the other hand, it follows from properties of Lie groups that the group $\mathrm{GL}(n, \mathbb{C})$ has the property that there is some neighborhood of the identity which contains no nontrivial subgroups. That is, $\mathrm{GL}(n, \mathbb{C})$ has *no small subgroups*. This may be used to obtain the following description of finite dimensional continuous representations of compact totally disconnected groups.

Proposition 1.1. *Let Γ be a compact totally disconnected group, and let*

$$\pi : \Gamma \to \mathrm{GL}(n, \mathbb{C})$$

be a homomorphism. Then π is continuous if and only if $\ker(\pi)$ is open in Γ.

Proof. See Problem set 4. $\square$

So, if (π, V) is a continuous finite dimensional representation of a compact totally disconnected group Γ, then $N = \ker(\pi)$ is an open normal subgroup of Γ, and is in particular of finite index, since Γ is compact. Thus, we may consider (π, V) to be a representation of the finite group Γ/N. For any compact totally disconnected group Γ (including finite groups), we let $\mathrm{Irr}(\Gamma)$ denote the collection of isomorphism classes of irreducible continuous finite dimensional representations (π, V) of Γ.

If G is a locally compact totally disconnected group, we define a *smooth representation* of G to be a pair (π, V), where V is any nonzero vector space over $\mathbb{C}$ (possibly infinite dimensional, but not necessarily with any topology), and $\pi : G \to \mathrm{Aut}(V)$ is a homomorphism such that the stabilizer in G of any vector $v \in V$ is open. Here, the stabilizer of $v \in V$ in G is the subgroup

$$\mathrm{stab}_G(v) = \{g \in G \mid \pi(g)v = v\} \subset G.$$

If (π, V) is a smooth representation of G, then we call it an *admissible representation* of G if, for every open subgroup $H \subset G$, the space of H-invariants in V,

$$V^H = \{v \in V \mid \pi(h)v = v \text{ for all } h \in H\} \subset V,$$

is a finite dimensional subspace of V. If (π, V) is a smooth representation of G, a subspace $W \subset V$ is called *G-invariant* if $\pi(g)w \in W$ for every $g \in G$ and $w \in W$. If W is a G-invariant subspace of V, then we have a homomorphism $\pi_W : G \to \mathrm{Aut}(W)$ defined by $\pi_W(g)w = \pi(g)w$. It is an exercise to check that if (π, V) is a smooth or admissible representation of G and W is a G-invariant subspace of V, then (π_W, W) is also a smooth or admissible representation of G, respectively. A smooth representation (π, V) of G is called *irreducible* if V contains no nonzero nontrivial G-invariant subspaces.

If (π_1, V_1) and (π_2, V_2) are two smooth representations of G, a linear transformation $T : V_1 \to V_2$ such that

$$T \circ \pi_1(g) = \pi_2(g) \circ T \quad \text{for all } g \in G$$

is called an *intertwining map* from (π_1, V_1) to (π_2, V_2). If there is an intertwining map T which is a bijection, then we say that (π_1, V_1) and (π_2, V_2) are *isomorphic* and T is an *isomorphism*, and we write $\pi_1 \cong \pi_2$, or $V_1 \cong V_2$. Note that if $\pi_1 \cong \pi_2$, then $\ker(\pi_1) = \ker(\pi_2)$.

Example 1. Let G be a compact totally disconnected group, and let (π, V) be a finite dimensional continuous representation of G. From Proposition 1.1, we know that $\ker(\pi)$ is open in G. For any $v \in V$, we have $\ker(\pi) \subset \mathrm{stab}_G(v)$. Since $\mathrm{stab}_G(v)$ is a subgroup of G which contains an open subgroup, then $\mathrm{stab}_G(v)$ is also open, since any element $g \in \mathrm{stab}_G(v)$ is contained in a coset of $\ker(\pi)$. Thus, (π, V) is smooth, and since it is assumed to be finite dimensional, then it is also admissible.

Example 2. Let F be a non-archimedean local field with absolute value $|\cdot|_v$ and ring of integers $\mathcal{O}$. Then $G = F^\times$ is a locally compact totally disconnected group. Let $V = \mathbb{C}^2$, and consider the homomorphism $\pi : F^\times \to \mathrm{GL}(2, \mathbb{C})$ given by

$$\pi(a) = \begin{pmatrix} 1 & \log|a|_v \\ 0 & 1 \end{pmatrix}.$$

If $v = \begin{pmatrix} x \\ y \end{pmatrix} \in V$, then for $y \neq 0$, we have

$$\mathrm{stab}_G(v) = \{a \in F^\times \mid \log|a|_v = 0\} = \mathcal{O}^\times,$$

which is open in G. For the case that $y = 0$, we have $\mathrm{stab}_G(v) = G$, which is of course open in G. So, (π, V) is smooth, and is admissible since V is finite dimensional.

We have noticed that the one-dimensional subspace

$$W = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \mid x \in \mathbb{C} \right\}$$

is invariant under all of G, and so (π, V) is not irreducible. Suppose that W' is some one-dimensional subspace of G, and let $w = \begin{pmatrix} x' \\ y' \end{pmatrix} \in W'$, with $w \neq 0$. Then, if W' is G-invariant, then we must have, for any $a \in G$, $\pi(a)w = \lambda w$ for some $\lambda \in \mathbb{C}$. We have

$$\pi(a)w = \begin{pmatrix} x' + y' \log|a|_v \\ y' \end{pmatrix},$$

and if this is to equal λw for some $\lambda \in \mathbb{C}$, then either $y' = 0$, in which case $W' = W$, or $\lambda = 1$, in which case the equality is not satisfied unless $a \in \mathcal{O}^\times$. In other words, W is the only G-invariant one-dimensional subspace of V. In particular, W has no G-invariant direct complement in V, and we see that reducible admissible representations are not necessarily completely reducible.

If G is a locally compact totally disconnected group, (π, V) is a smooth representation of G, and H is a closed subgroup of G (and so locally compact totally disconnected), then let $(\pi|_H, V)$ denote the representation of H obtained by restricting π to H. It is immediate that $(\pi|_H, V)$ is smooth, since for any $v \in V$ we have

$$\mathrm{stab}_H(v) = H \cap \mathrm{stab}_G(v),$$

which is open in H. If K is a compact totally disconnected group, (π, V) is a smooth representation of K, and (ρ, W_ρ) is an irreducible finite dimensional continuous representation of K, or $\rho \in \mathrm{Irr}(K)$, then the *ρ-isotypic part* of (π, V), denoted $V(\rho)$, is the sum of all K-invariant subspaces W of V which are isomorphic to W_ρ as smooth representations, so $\pi_W \cong \rho$. That is,

$$V(\rho) = \sum_{\substack{W \subset V \\ W \cong V_\rho}} W.$$

Recall that in the case that Γ is a finite group, and (π, V) is any representation of Γ (not necessarily finite dimensional), then we have

$$V = \bigoplus_{\rho \in \mathrm{Irr}(\Gamma)} V(\rho).$$

We have a similar situation when we restrict a smooth representation of a locally compact totally disconnected group to a compact open subgroup, as we see now.

Theorem 1.1. *Let G be a locally compact totally disconnected group, and let K be a compact open subgroup of G. If (π, V) is a smooth representation of G, then*

$$V = \bigoplus_{\rho \in \mathrm{Irr}(K)} V(\rho), \tag{1.1}$$

where $V(\rho)$ is the ρ-isotypic part of $(\pi|_K, V)$.

The representation (π, V) of G is admissible if and only if each $V(\rho)$ in (1.1) is finite dimensional. In this case, the sum (1.1) is a direct sum of irreducible finite dimensional continuous representations of K, and thus $(\pi|_K, V)$ is completely reducible.

Proof. We know that $\sum_{\rho \in \mathrm{Irr}(K)} V(\rho) \subset V$, and we now show the other containment. Let $v \in V$. Since (π, V) is smooth, then $H = \mathrm{stab}_G(v)$ is open in G. Then $H' = H \cap K$ is a compact open subgroup in K which also stabilizes v. Since K is a compact totally disconnected group, then we can find a compact open normal subgroup K_0 which is contained in H', so that K_0 stabilizes v, and $v \in V^{K_0}$. Let $k \in K$, $h \in K_0$, and $w \in V^{K_0}$. Since K_0 is normal in K, then $hk = kh'$ for some $h' \in K_0$. This implies that V^{K_0} is a K-invariant subspace of V, since we have

$$\pi(h)\pi(k)w = \pi(k)\pi(h')w = \pi(k)w,$$

so that $w \in V^{K_0}$ and $k \in K$ implies that $\pi(k)w \in V^{K_0}$. Since K is compact, K_0 must have finite index in K, so $\Gamma = K/K_0$ is a finite group, and we may consider V^{K_0} to be a representation space for Γ with the action of $\pi|_K$. In particular, any representation of Γ may be considered a representation on K with kernel containing K_0. So, we have

$$v \in V^{K_0} = \bigoplus_{\rho \in \mathrm{Irr}(\Gamma)} V^{K_0}(\rho) \subset \sum_{\rho \in \mathrm{Irr}(K)} V(\rho).$$

Now we have $V = \sum_{\rho \in \mathrm{Irr}(K)} V(\rho)$, and we must show that the sum is direct. If it is not, then there is some finite collection $\{\rho_1, \ldots, \rho_m\} \subset \mathrm{Irr}(K)$ such that $\sum_{i=1}^m c_i v_i = 0$, where $v_i \in V(\rho_i)$, and the c_i are scalars in $\mathbb{C}$ which are not all 0. Each $\ker(\rho_i)$ is open by

Proposition 1.1, and we let $\tilde{K} = \cap_{i=1}^m \ker(\rho_i)$, which is also open. By the same argument as before, $V^{\tilde{K}}$ is a K-invariant subspace of V, and letting $\tilde{\Gamma} = K/\tilde{K}$, we have

$$V^{\tilde{K}} = \bigoplus_{\tilde{\rho} \in \mathrm{Irr}(\tilde{\Gamma})} V^{\tilde{K}}(\tilde{\rho}). \tag{1.2}$$

Since $\tilde{K} \subset \ker(\rho_i)$ for each i, each ρ_i may be considered as a representation of $\tilde{\Gamma}$, that is each $\rho_i \in \mathrm{Irr}(\tilde{\Gamma})$. In particular, we have $v_i \in V^{\tilde{K}}(\rho_i)$, and the assumption that the v_i are linearly dependent contradicts the direct sum in (1.2). Thus, we have (1.1).

For the second part, suppose that (π, V) is admissible. For any $(\rho, V_\rho) \in \mathrm{Irr}(K)$, let $W \subset V$ be a K-invariant subspace of V with σ the representation $\pi|_K$ acting on W, and suppose $\sigma \cong \rho$ as representations of K. Then $\ker(\sigma) = \ker(\rho)$ is open in K by Proposition 1.1, and so open in G. Since (π, V) is admissible, then $V^{\ker(\sigma)}$ is finite dimensional, and $W \subset V^{\ker(\sigma)} = V^{\ker(\rho)}$, since W is the space associated with σ. This is true for any K-invariant subspace $W \subset V$ with associated K-representation isomorphic to ρ, and so we have $V(\rho) \subset V^{\ker(\rho)}$, and thus $V(\rho)$ is finite dimensional. Conversely, if (π, V) is not admissible, then V^H is infinite dimensional for some open subgroup H of G. Then $H \cap K$ is a compact open subgroup of G, and letting K_0 be a compact open normal subgroup in $H \cap K$, we have V^{K_0} is infinite dimensional since it contains V^H. Now, $\Gamma_0 = K/K_0$ is a finite group, and V^{K_0} is a K-invariant subspace, and so we have

$$V^{K_0} = \bigoplus_{\rho \in \mathrm{Irr}(\Gamma_0)} V^{K_0}(\rho).$$

Since $\mathrm{Irr}(\Gamma_0)$ is finite, and V^{K_0} is infinite dimensional, then some $V^{K_0}(\rho)$ is infinite dimensional. But each $\rho \in \mathrm{Irr}(\Gamma_0)$ is an irreducible representation of K with K_0 contained in its kernel, and so $V^{K_0}(\rho) \subset V(\rho)$, so that we also have some $V(\rho)$ which is infinite dimensional.

For the final statement, we may assume that the representation (π, V) is such that $V = V(\rho)$ for some $\rho \in \mathrm{Irr}(K)$, where we know V here is finite dimensional since (π, V) is assumed admissible. For any K-invariant subspace W of $V = V(\rho)$, such that $\sigma \cong \rho$, where σ is $\pi|_K$ acting on W, we have $\ker(\sigma) = \ker(\rho)$. We are assuming that V is the sum of all such subspaces, and so we have $\ker(\pi|_K) = \ker(\rho)$. From Proposition 1.1, we have $\ker(\rho) = \ker(\pi|_K)$ is open, and so $\pi|_K$ is continuous. The result now follows from Maschke's Theorem for finite dimensional continuous representations of compact groups. $\square$

We may apply Theorem 1.1 to completely understand the smooth and admissible representations of compact totally disconnected groups.

Corollary 1.1. *Let K be a compact totally disconnected group. If V is a finite dimensional $\mathbb{C}$-vector space, and $\pi : K \to \mathrm{GL}(V)$ is a homomorphism, then (π, V) is a smooth representation of K if and only if it is a continuous representation of K. The only irreducible admissible representations of K are finite dimensional continuous representations.*

Proof. We have already seen that Proposition 1.1 implies that every finite dimensional continuous representation of K is smooth. So, we assume (π, V) is smooth, and since V is assumed to have finite dimension, then (π, V) is admissible. Since K is a compact open subgroup of itself and V is finite dimensional, then (π, V) is a finite direct sum of irreducible continuous representations of K. A direct sum of continuous representations is continuous, then (π, V) is continuous. For the second statement, if (π, V) is an irreducible admissible representation of K, then V has no nonzero K-invariant subspaces other than V itself, and the direct sum decomposition from Theorem 1.1 can have only one summand. Thus (π, V) is an irreducible finite dimensional continuous representation of K. □

Although the space for an admissible representation may be infinite dimensional, the finiteness condition that comes with admissibility is enough to give us the following.

Proposition 1.2 (Schur's Lemma). *Let G be a locally compact totally disconnected group, and let (π, V) be an irreducible admissible representations of G. If $T : V \to V$ is an isomorphism from (π, V) to itself, then there is a scalar $\lambda \in \mathbb{C}$ such that $T(v) = \lambda v$ for all $v \in V$.*

Proof. Let $w \in V$ be a nonzero vector in V. Since (π, V) is smooth, then $H_0 = \mathrm{stab}_G(w)$ is open in G. Now, V^{H_0} is finite dimensional since (π, V) is admissible. We have $w \in V^{H_0}$, and in particular V^{K_0} is nonzero. For any $x \in H_0$ and $v_0 \in V^{H_0}$, we have $\pi(x)(T(v_0)) = T(\pi(x)v_0) = T(v_0)$, and so $T(v_0) \in V^{H_0}$, and T preserves the space V^{H_0}. Let $T_0 : V^{H_0} \to V^{H_0}$ be the restriction of T to V^{H_0}, and we have T_0 has some eigenvalue $\lambda \in \mathbb{C}$ since V^{H_0} is a finite dimensional $\mathbb{C}$-vector space. Letting $I : V \to V$ be the identity transformation, we have $T - \lambda I$ has a nonzero kernel, in particular it contains the eigenspace of λ for T_0 in V^{H_0}. Now, $W = \ker(T - \lambda I)$ is G-invariant, since if $w_0 \in W$, then $T(w_0) = \lambda w_0$, and for any $g \in G$, we have

$$(T - \lambda I)(\pi(g)w_0) = (T \circ \pi(g))w_0 - \pi(g)\lambda w_0 = (\pi(g) \circ T)w_0 - \pi(g)\lambda w_0 = 0.$$

But we have assumed that (π, V) is irreducible, and so we must have $V = \ker(T - \lambda I)$, and so $T(v) = \lambda v$ for all $v \in V$. □

2. The Hecke algebra

In this section, G will be a unimodular locally compact totally disconnected group. Fix a Haar measure on G denoted by μ or $\int dh$. A function $f : G \to \mathbb{C}$ is *smooth* if it is locally constant, that is, for each $g \in G$ there is an open neighborhood U of g such that $f(x) = f(g)$ for every $x \in U$. Note that any smooth function on G is in particular continuous. Let $C_c^\infty(G)$ denote the set of smooth functions on G with compact support. If $f \in C_c^\infty(G)$, then let $Y = \mathrm{supp}(f)$, and for each $y \in Y$, choose an open neighborhood U_y of y on which f is constant. Since Y is compact, there is a finite collection of these U_y, say $\{U_i\}_{i=1}^m$, such that $Y = \cup_{i=1}^m U_i$. We may assume that the U_i are disjoint, and that f takes a different value on each U_i. Then we see that we have $f = \sum_{i=1}^m c_i \mathbf{1}_{U_i}$, where $\mathbf{1}_{U_i}$ is the indicator function for U_i, and $c_i = f(y_i)$ for any $y_i \in U_i$. Notice that each U_i is necessarily compact, since $\mathrm{supp}(f)$ is compact, and is the disjoint union of the U_i.

Now let $f_1, f_2 \in C_c^\infty(G)$, and consider the convolution $f_1 * f_2$, given by

$$(f_1 * f_2)(g) = \int_G f_1(gh^{-1}) f_2(h)\, dh.$$

Suppose that $Y_1 = \mathrm{supp}(f_1) = \cup_{i=1}^m U_i$, and $Y_2 = \mathrm{supp}(f_2) = \cup_{j=1}^k V_j$, where the distinct nonzero values of f_1 and f_2 are taken on the U_i and V_j, respectively. It may be directly checked that $\mathrm{supp}(f_1 * f_2) \subset Y_1 Y_2 = \cup_{i,j} U_i V_j$, and $f_1 * f_2$ is locally constant, so that $f_1 * f_2 \in C_c^\infty(G)$.

We define *the Hecke algebra* of G, denote $\mathcal{H}(G)$, to be the algebra over $\mathbb{C}$ with elements from $C_c^\infty(G)$, addition given by pointwise addition, and multiplication given by convolution. For any compact open subgroup K_0 in G, we let $\mathcal{H}_{K_0}(G)$ to be the subset of $\mathcal{H}(G)$ consisting of functions which are bi-invariant under multiplication by K_0, that is,

$$\mathcal{H}_{K_0}(G) = \{\phi \in \mathcal{H}(G) \mid \phi(k_1 g k_2) = \phi(g) \text{ for all } k_1, k_2 \in K_0\}.$$

For any compact open subset S in G, define $\epsilon_S = \mu(S)^{-1}\mathbf{1}_S \in \mathcal{H}(G)$. Then $\epsilon_{K_0} = \mu(K_0)^{-1}\mathbf{1}_{K_0}$, and we have $\epsilon_{K_0} \in \mathcal{H}_{K_0}(G)$.

Proposition 2.1. *Let G be a unimodular locally compact totally disconnected group, and let K_0 be a compact open subgroup of G.*

(1) *For every $\phi \in \mathcal{H}_{K_0}(G)$, $\epsilon_{K_0} * \phi = \phi * \epsilon_{K_0} = \phi$.*
(2) *$\mathcal{H}_{K_0}(G) = \epsilon_{K_0}\mathcal{H}(G)\epsilon_{K_0}$, where $\epsilon_{K_0}\mathcal{H}(G)\epsilon_{K_0} = \{\epsilon_{K_0} * f * \epsilon_{K_0} \mid f \in \mathcal{H}(G)\}$.*
(3) *$\mathcal{H}_{K_0}(G)$ is a $\mathbb{C}$-algebra with unit ϵ_{K_0}, and is a subalgebra of $\mathcal{H}(G)$.*

Proof. (1): If $\phi \in \mathcal{H}_{K_0}(G)$, and $g \in G$, we have

$$(\phi * \epsilon_{K_0})(g) = \int_G \phi(gh^{-1})\epsilon_{K_0}(h)\,dh = \mu(K_0)^{-1}\int_{K_0} \phi(gh^{-1})\,dh = \mu(K_0)^{-1}\int_{K_0}\phi(g)\,dh,$$

the last equality coming from the fact that $\phi \in \mathcal{H}_{K_0}(G)$. The last integral is just $\phi(g)$, and so $\phi * \epsilon_{K_0} = \phi$. We also have

$$(\epsilon_{K_0} * \phi)(g) = \int_G \epsilon_{K_0}(gh^{-1})\phi(h)\,dh.$$

Since $\int dh$ is a bi-invariant Haar measure, we may make the changes of variables $h \mapsto h^{-1}$, and then $h \mapsto g^{-1}h$, and we have

$$(\epsilon_{K_0} * \phi)(g) = \int_G \epsilon_{K_0}(h)\phi(h^{-1}g)\,dh = \mu(K_0)^{-1}\int_{K_0}\phi(h^{-1}g)\,dh = \phi(g),$$

as in the previous case.

(2): First, if $\phi \in \mathcal{H}_{K_0}(G)$, then from part (1), we have $\phi = \epsilon_{K_0} * \phi * \epsilon_{K_0}$, and so $\mathcal{H}_{K_0}(G) \subset \epsilon_{K_0}\mathcal{H}(G)\epsilon_{K_0}$. Now let $f \in \mathcal{H}(G)$ and $g \in G$. Then

$$\begin{aligned}(\epsilon_{K_0} * f * \epsilon_{K_0})(g) &= \int_G \epsilon_{K_0}(gh_2^{-1})(f * \epsilon_{K_0})(h_2)\,dh_2 \\ &= \int_G\int_G \epsilon_{K_0}(gh_2^{-1})f(h_2h_1^{-1})\epsilon_{K_0}(h_1)\,dh_1\,dh_2. \qquad (2.1)\end{aligned}$$

So, for $k_1, k_2 \in K_0$, we have

$$(\epsilon_{K_0} * f * \epsilon_{K_0})(k_1gk_2) = \int_G\int_G \epsilon_{K_0}(k_1gk_2h_2^{-1})f(h_2h_1^{-1})\epsilon_{K_0}(h_1)\,dh_1\,dh_2.$$

Making the changes of variables $h_2 \mapsto h_2k_2$ and $h_1 \mapsto h_1k_2$ gives

$$(\epsilon_{K_0} * f * \epsilon_{K_0})(k_1gk_2) = \int_G\int_G \epsilon_{K_0}(k_1gh_2^{-1})f(h_2h_1^{-1})\epsilon_{K_0}(h_1k_2)\,dh_1\,dh_2.$$

Since $\epsilon_{K_0} \in \mathcal{H}_{K_0}(G)$, we have $\epsilon_{K_0}(k_1gh_2^{-1}) = \epsilon_{K_0}(gh_2^{-1})$ and $\epsilon_{K_0}(h_1k_2) = \epsilon_{K_0}(h_1)$, and the inegral simplifies to (2.1). Thus $\epsilon_{K_0} * f * \epsilon_{K_0} \in \mathcal{H}_{K_0}(G)$, and we have $\mathcal{H}_{K_0}(G) = \epsilon_{K_0}\mathcal{H}(G)\epsilon_{K_0}$.

(3): To check that $\mathcal{H}_{K_0}(G)$ is a $\mathbb{C}$-algebra, and thus a subalgebra of $\mathcal{H}(G)$, it is enough to check that it is closed under multiplication. If $\phi_1, \phi_2 \in \mathcal{H}_{K_0}(G)$, then from (1), we have $\phi_i = \epsilon_{K_0} * \phi_i = \phi_i * \epsilon_{K_0}$ for $i = 1, 2$. Then we have

$$\phi_1 * \phi_2 = \epsilon_{K_0} * (\phi_1 * \phi_2) * \epsilon_{K_0} \in \epsilon_{K_0}\mathcal{H}(G)\epsilon_{K_0} = \mathcal{H}_{K_0}(G),$$

from (2). The fact that ϵ_{K_0} is a unit for $\mathcal{H}_{K_0}(G)$ is exactly part (1). $\square$

So, for any compact open subgroup K_0 of G, we have $\epsilon_{K_0} * \epsilon_{K_0} = \epsilon_{K_0}$, and so each ϵ_{K_0} is an idempotent in $\mathcal{H}(G)$. Also, if K_0 and K_1 are compact open subgroups of G such that $K_1 \subset K_0$, then $\mathcal{H}_{K_0}(G) \subset \mathcal{H}_{K_1}(G)$, and $\epsilon_{K_0} * \epsilon_{K_1} = \epsilon_{K_0}$. We see now that every element of $\mathcal{H}(G)$ is in one of the algebras $\mathcal{H}_{K_0}(G)$.

Proposition 2.2. *For any unimodular locally compact totally disconnected group G,*

$$\mathcal{H}(G) = \bigcup_{K_0 \subset G} \mathcal{H}_{K_0}(G),$$

where the union runs over all compact open subgroups K_0 of G.

Proof. Let $f \in \mathcal{H}(G)$, and say $f = \sum_{i=1}^m c_i \mathbf{1}_{U_i}$, for pairwise disjoint compact open sets U_i, and the c_i are distinct, where $c_i = f(x_i)$ for any $x_i \in U_i$. For each U_i, and each $x \in U_i$, let $K_x^{(i)}$ be a compact open subgroup of G such that $K_x^{(i)} \subset x^{-1}U_i$, so U_i is covered by all of the $xK_x^{(i)}$. Since U_i is compact, take a finite subcover $x_j K_{x_j}^{(i)}$, for $j = 1, \dots k$. Now let $K^{(i)} = \cap_{j=1}^k K_{x_j}^{(i)}$, and let $K_1 = \cap_{i=1}^m K^{(i)}$. We repeat this process, except for each U_i and each $y \in U_i$, we let $K_y^{(i)'}$ be a compact open subgroup of G such that $K_y^{(i)'} \subset U_i y^{-1}$. Using the same subcover argument and notation, we let $K_2 = \cap_{i=1}^m K^{(i)'}$, and we finally let $K_0 = K_1 \cap K_2$. Notice that K_0 has the property that any of its right or left cosets is either completely contained in some U_i, or is disjoint from all of them. It follows that for any $k_1, k_2 \in K_0$, and $x \in G$, we have $x \in U_i$ if and only if $k_1 x k_2 \in U_i$. Since f is constant on each U_i, we now have $f \in \mathcal{H}_{K_0}(G)$. $\square$

Now suppose that (π, V) is a smooth representation of G. For any $f \in \mathcal{H}(G)$ and $v \in V$, define $\pi(f)v$ by

$$\pi(f)v = \int_G f(g)\pi(g)v\, dg.$$

We note that we may write the above integral as a finite sum, as follows. For $v \in V$, we know that $H_0 = \mathrm{stab}_G(v)$ is open in G since (π, V) is smooth. Given $f \in \mathcal{H}(G)$, choose K_0 such that $f \in \mathcal{H}_{K_0}(G)$ as given in Proposition 2.2, and let $\tilde{K} = K_0 \cap H_0$. Now, $\tilde{K}$ stabilizes v, and the support of f may be written as a union of left cosets of $\tilde{K}$, say $\mathrm{supp}(f) = \cup_{j=1}^m g_j\tilde{K}$, so that f is constant on each $g_j\tilde{K}$. Now we have

$$\pi(f)v = \int_G f(g)\pi(g)v\, dg = \sum_{j=1}^m \int_{g_j\tilde{K}} f(g)\pi(g)v\, dg = \mu(\tilde{K}) \sum_{j=1}^m f(g_j)\pi(g_j)v. \tag{2.2}$$

We may directly check that for any $f_1, f_2 \in \mathcal{H}(G)$ and $v \in V$, we have

$$\pi(f_1 * f_2)v = \pi(f_1)(\pi(f_2)v),$$

so that π defines a ring homomorphism $\pi : \mathcal{H}(G) \to \mathrm{End}(V)$. That is, V may be viewed as an $\mathcal{H}(G)$-module by defining the action $f \cdot v = \pi(f)v$.

Proposition 2.3. *Let G be a unimodular locally compact totally disconnected group with (π, V) a smooth representation, and K_0 a compact open subgroup of G. Then $V^{K_0} = \epsilon_{K_0} \cdot V$, where $\epsilon_{K_0} \cdot V = \{\pi(\epsilon_{K_0})v \mid v \in V\}$.*

Proof. If $v \in V$, then

$$\pi(\epsilon_{K_0})v = \int_G \epsilon_{K_0}(g)\pi(g)v\,dg = \int_{K_0} \mu(K_0)^{-1}\pi(g)v\,dg.$$

For any $k \in K_0$, we have

$$\pi(k)(\pi(\epsilon_{K_0})v) = \pi(k)\int_{K_0} \mu(K_0)^{-1}\pi(g)v\,dg = \int_{K_0} \mu(K_0)^{-1}\pi(kg)v\,dg,$$

and the change of variables $g \mapsto k^{-1}g$ gives $\pi(k)(\pi(\epsilon_{K_0})v) = \pi(\epsilon_{K_0})v$. Thus $\epsilon_{K_0} \cdot V \subset V^{K_0}$. For the opposite containment, let $v \in V^{K_0}$ so that $\pi(k)v = v$ for every $k \in K_0$. Then

$$\pi(\epsilon_{K_0})v = \int_{K_0} \mu(K_0)^{-1}\pi(g)v\,dg = \int_{K_0} \mu(K_0)^{-1}v\,dg = v.$$

So, $v = \pi(\epsilon_{K_0})v \in \epsilon_{K_0} \cdot V$, and we have $V^{K_0} = \epsilon_{K_0} \cdot V$. $\square$

Similar to representations of a finite group corresponding to modules over the group algebra, and properties of these representations coming from the fact that the group algebra is semisimple, we may gain knowledge about smooth representations by studying $\mathcal{H}(G)$-modules in a more general context.

Let Ω be an algebraically closed field. An *idempotented algebra* over Ω is a pair $(\mathcal{A}, \mathcal{E})$, where $\mathcal{A}$ is an Ω-algebra (not necessarily with unit), and $\mathcal{E} \subset \mathcal{A}$ is a collection of idempotent elements of $\mathcal{A}$ with the following properties:

(1) If $e_1, e_2 \in \mathcal{E}$, then there exists an $e_0 \in \mathcal{E}$ such that $e_0 e_i = e_i$ for $i = 1, 2$.
(2) For every element $\phi \in \mathcal{A}$, there is an $e \in \mathcal{E}$ such that $e\phi = \phi e = \phi$.

For every $e \in \mathcal{E}$, note that $e\mathcal{A}e$ is an Ω-algebra with unit e. If M is an $\mathcal{A}$-module, then eM is an $e\mathcal{A}e$-module. We call an $\mathcal{A}$-module M *smooth* if $M = \cup_{e \in \mathcal{E}} eM$, and *admissible* if eM is finite dimensional as a vector space over Ω for every $e \in \mathcal{E}$.

In the case that G is a unimodular locally compact totally disconnected group, consider the $\mathbb{C}$-algebra $\mathcal{H}(G)$, with the set of idempotents $\mathcal{E} = \{\epsilon_{K_0} \mid K_0 \subset G\}$, where K_0 runs over all compact open subgroups of G. If K_1 and K_2 are compact open subgroups of G, then if $K_0 = K_1 \cap K_2$, we have $\epsilon_{K_0} * \epsilon_{K_i} = \epsilon_{K_i}$ for $i = 1, 2$. Also, if $f \in \mathcal{H}(G)$, then from Proposition 2.2, $f \in \mathcal{H}_{K_0}(G)$ for some compact open subgroup K_0, and so $f * \epsilon_{K_0} = \epsilon_{K_0} * f = f$ from Proposition 2.1(1). Thus, $(\mathcal{H}(G), \mathcal{E})$ is an idempotented algebra over $\mathbb{C}$.

Proposition 2.4. *Let (π, V) be a smooth representation of G. Then:*

(1) *V is a smooth $\mathcal{H}(G)$-module, and is an admissible $\mathcal{H}(G)$-module if (π, V) is an admissible representation.*
(2) *V is simple as an $\mathcal{H}(G)$-module if and only if (π, V) is an irreducible representation of G.*

Proof. (1): For any $v \in V$, we know $\mathrm{stab}_G(v)$ is an open subgroup of G, and we let K_0 be a compact open subgroup inside of it, so that $v \in V^{K_0}$. So, we have $V = \cup V^{K_0}$, where the union runs over all compact open subgroups of G, and from Proposition 2.3,

$V^{K_0} = \epsilon_{K_0} \cdot V$, and V is a smooth $\mathcal{H}(G)$-module by definition. If (π, V) is admissible, then each $V^{K_0} = \epsilon_{K_0} \cdot V$ is finite dimensional, and V is an admissible $\mathcal{H}(G)$-module.

(2): Suppose $W \subset V$ is a nonzero proper G-invariant subspace of V. Then for any $f \in \mathcal{H}(G)$ and $w \in W$, it follows from the finite sum version for $\pi(f)w$ in (2.2) that $\pi(f)w \in W$, and W is an $\mathcal{H}(G)$-submodule of V.

Conversely, suppose that W is a nonzero proper $\mathcal{H}(G)$-submodule of V, and let $g \in G$, $w \in W$. Since (π, V) is smooth, $\mathrm{stab}_G(w)$ is open, and we let K_0 be a compact open subgroup contained in it. Now,

$$\pi(\epsilon_{gK_0})w = \int_{gK_0} \mu(gK_0)^{-1}\pi(h)w\,dh = \pi(g)w.$$

Since $\epsilon_{gK_0} \in \mathcal{H}(G)$, then we must have $\pi(g)w \in W$, and so W is a nonzero proper G-invariant subspace of V. □

In the proof of Proposition 2.4(2), we saw that if (π, V) is a smooth representation of G, and if $w \in V^{K_0}$ for some compact open subgroup K_0 of G, then $\pi(\epsilon_{gK_0})w = \pi(g)w$. Given any smooth $\mathcal{H}(G)$-module V, one may take advantage of this observation to define a smooth representation (π, V) of G such that $\pi(f)v = f \cdot v$ for every $f \in \mathcal{H}(G)$, $v \in V$. That is, all smooth representations come from smooth $\mathcal{H}(G)$-modules.

Proposition 2.5. *Let V be a smooth $\mathcal{H}(G)$-module. Then there exists a smooth representation (π, V) of G such that $\pi(f)v = f \cdot v$ for all $f \in \mathcal{H}(G)$, $v \in V$.*

Proof. See Problem set 4. □

Let (π, V) be a smooth representation of G and K_0 a compact open subgroup of G. Since $\mathcal{H}_{K_0}(G) = \epsilon_{K_0}\mathcal{H}(G)\epsilon_{K_0}$ (by Proposition 2.1(2)) and $V^{K_0} = \epsilon_{K_0} \cdot V$ (by Proposition 2.3), then V^{K_0} is an $\mathcal{H}_{K_0}(G)$-module. Understanding V^{K_0} as an $\mathcal{H}_{K_0}(G)$-module for every compact open subgroup K_0 of G is essentially the same as understanding V as an $\mathcal{H}(G)$-module, as the next several results show.

Proposition 2.6. *Let (π, V) be a smooth representation of G. Then V is a simple $\mathcal{H}(G)$-module if and only if V^{K_0} is either* 0 *or simple as an $\mathcal{H}_{K_0}(G)$-module for every compact open subgroup K_0 of G.*

Proof. Let K_0 be a compact open subgroup of G, and suppose that $W \subset V^{K_0}$ is a proper nonzero $\mathcal{H}_{K_0}(G)$-submodule of V^{K_0}, where V^{K_0} is assumed to be nonzero. Consider $\mathcal{H}(G)W$, where

$$\mathcal{H}(G)W = \left\{ \sum_{i=1}^{k} \pi(f_i)w_i \;\middle|\; k \in \mathbb{Z}_{>0}, f_i \in \mathcal{H}(G), w_i \in W \right\}.$$

Note that $\mathcal{H}(G)W$ is an $\mathcal{H}(G)$-submodule of V. Suppose that $v_0 \in \mathcal{H}(G) \cap V^{K_0}$, where $v_0 = \sum_{i=1}^{k} \pi(f_i)w_i$. We have $\pi(\epsilon_{K_0})w_i = w_i$ for each w_i, since $W \subset V^{K_0} = \epsilon_{K_0} \cdot V$, and $\pi(\epsilon_{K_0})v_0 = v_0$ since we are assuming $v_0 \in V^{K_0}$. Thus, we have

$$v_0 = \pi(\epsilon_{K_0})v_0 = \pi(\epsilon_{K_0})\sum_{i=1}^{k}\pi(f_i)\pi(\epsilon_{K_0}w_i = \sum_{i=1}^{k}\pi(\epsilon_{K_0} * f_i * \epsilon_{K_0})w_i \in W,$$

since $\epsilon_{K_0} * f_i * \epsilon_{K_0} \in \mathcal{H}_{K_0}(G)$, and W is an $\mathcal{H}_{K_0}$-module. Now we have $\mathcal{H}(G)W \cap V^{K_0} \subset W$, but we also have the opposite containment, and so $\mathcal{H}(G)W \cap V^{K_0} = W$. In particular,

$\mathcal{H}(G)W$ is nonzero, and is a proper $\mathcal{H}(G)$-submodule of V, otherwise $\mathcal{H}(G)W \cap V^{K_0}$ would be all of V^{K_0}. Therefore, if V^{K_0} is nonzero and not a simple $\mathcal{H}_{K_0}(G)$-module, then V is not a simple $\mathcal{H}(G)$-module.

Conversely, suppose that W is a nonzero proper $\mathcal{H}(G)$-submodule of V. For any compact open subgroup K_0 of G, we have $W^{K_0} \subset V^{K_0}$. We have $V = \cup V^{K_0}$, and since (π_W, W) is a smooth representation, we also have $W = \cup W^{K_0}$, where the unions range over all compact open subgroups K_0 of G. Choose a compact open subgroup K_1 of G such that $W^{K_1} \neq 0$. If W^{K_1} is properly contained in V^{K_1}, then it is a proper nonzero $\mathcal{H}_{K_1}(G)$-submodule of V^{K_1}, and we are done. Otherwise, notice that we have $V = \cup_{K_0 \subset K_1} V^{K_0}$, since for any compact open subgroup K_2, if $\tilde{K} = K_2 \cap K_1$, then $V^{K_2} \subset V^{\tilde{K}}$, and $\tilde{K} \subset K_1$, and similarly, we have $W = \cup_{K_0 \subset K_1} W^{K_0}$. Now, there must be a compact open subgroup $K_0 \subset K_1$ such that W^{K_0} is properly contained in V^{K_0}, otherwise we would have $V = W$, contradicting the fact that W is properly contained in V. Now we have V^{K_0} is not a simple $\mathcal{H}_{K_0}(G)$-module. □

We need the following variant of Schur's Lemma.

Lemma 2.1. *Let (π_1, V_1) and (π_2, V_2) be irreducible admissible representations of G, and K_0 a compact open subgroup of G such that $V_1^{K_0} \cong V_2^{K_0}$ as $\mathcal{H}_{K_0}(G)$-modules, and are both nonzero. Then an $\mathcal{H}_{K_0}(G)$-module isomorphism $\sigma_0 : V_1^{K_0} \to V_2^{K_0}$ is determined up to scalar multiple.*

Proof. Let $\sigma : V_1^{K_0} \to V_1^{K_0}$ be an $\mathcal{H}_{K_0}(G)$-module isomorphism from $V_1^{K_0}$ to itself. Since (π_1, V_1) is assumed to be admissible, then $V_1^{K_0}$ is a finite dimensional vector space, and σ is a linear transformation since for $c \in \mathbb{C}$, $v \in V_1^{K_0}$, we have $c\sigma(v) = c\epsilon_{K_0} \cdot \sigma(v) = \sigma(c\epsilon_{K_0} \cdot v) = \sigma(cv)$. So, σ must have an eigenvalue $\lambda \in \mathbb{C}$, and if I is the identity transformation on V^{K_0}, then $W = \ker(\sigma - \lambda I) \subset V_1^{K_0}$ is nonzero. Now let $w \in W$ and $\phi \in \mathcal{H}_{K_0}(G)$, so that $\sigma(\phi \cdot w) = \phi \cdot \sigma(w)$ and $\sigma(w) = \lambda(w)$. We have

$$(\sigma - \lambda I)(\phi \cdot w) = \phi \cdot \sigma(w) - \phi \cdot \lambda w = 0.$$

So, if $W = \ker(\sigma - \lambda I)$ is an $\mathcal{H}_{K_0}(G)$-submodule of $V_1^{K_0}$. Since (π_1, V_1) is assumed irreducible and $V_1^{K_0}$ is nonzero, then by Propositions 2.4(2) and 2.6, $V_1^{K_0}$ is a simple $\mathcal{H}_{K_0}(G)$-module, and so $W = V_1^{K_0}$, and $\sigma(v) = \lambda v$ for every $v \in V_1^{K_0}$.

Now let $\sigma_0, \sigma_1 : V_1^{K_0} \to V_2^{K_0}$ be two $\mathcal{H}_{K_0}(G)$-module isomorphisms. Then $\sigma_0^{-1} \circ \sigma_1 : V_1^{K_0} \to V_1^{K_0}$ is an $\mathcal{H}_{K_0}(G)$-module isomorphism, and so there is some $\lambda \in \mathbb{C}$ such that $(\sigma_0^{-1} \circ \sigma_1)(v) = \lambda v$ for every $v \in V_1^{K_0}$, and thus $\sigma_1 = \lambda \sigma_0$. □

The next result gets to the main point, which is that isomorphisms at the level of $\mathcal{H}_{K_0}(G)$-modules for every compact open subgroup K_0 gives an isomorphism of $\mathcal{H}(G)$-modules.

Proposition 2.7. *Let (π_1, V_1) and (π_2, V_2) be irreducible admissible representations of G. If $V_1^{K_0} \cong V_2^{K_0}$ as $\mathcal{H}_{K_0}(G)$-modules for every compact open subgroup K_0 of G, then $\pi_1 \cong \pi_2$.*

Proof. Fix a compact open subgroup K_0 of G such that $V_1^{K_0}$ and $V_2^{K_0}$ are both nonzero, and fix an $\mathcal{H}_{K_0}(G)$-module isomorphism $\sigma_0 : V_1^{K_0} \to V_2^{K_0}$, which is determined up to scalar multiple by Lemma 2.1.

Let K_1 be another compact open subgroup of G such that $K_1 \subset K_0$, so that $\mathcal{H}_{K_0}(G) \subset \mathcal{H}_{K_1}(G)$. For $i = 1, 2$, we have $V_i^{K_0} = \epsilon_{K_0} \cdot V_i^{K_1}$, since $\epsilon_{K_0} \cdot V_i^{K_1} \subset \epsilon_{K_0} \cdot V_i = V_i^{K_0}$, and $V_i^{K_0} \subset V_i^{K_1}$, so that $V_i^{K_0} = \epsilon_{K_0} \cdot V_i^{K_0} \subset \epsilon_{K_0} \cdot V_i^{K_1}$.

We claim that σ_0 can be extended uniquely to an $\mathcal{H}_{K_1}(G)$-isomorphism $\sigma_{K_1} : V_1^{K_1} \to V_2^{K_2}$. We are assuming that there is some isomorphism σ_{K_1} from $V_1^{K_1}$ to $V_2^{K_2}$ under hypothesis. We have

$$\sigma_{K_1}(V_1^{K_0}) = \sigma_{K_1}(\epsilon_{K_0} \cdot V_1^{K_1}) = \epsilon_{K_0} \cdot \sigma_{K_1}(V_1^{K_1}) = \epsilon_{K_0} \cdot V_2^{K_1} = V_2^{K_0},$$

so that restricting σ_{K_1} to $V_1^{K_0}$ induces an $\mathcal{H}_{K_0}(G)$-module isomorphism from $V_1^{K_0}$ to $V_2^{K_0}$. By Lemma 2.1, we must then have that $\sigma_{K_1}|_{V_1^{K_0}}$ is a scalar multiple of σ_0. We thus choose σ_{K_1}, which is also determined up to scalar multiple, so that its restriction to $V_1^{K_0}$ is exactly σ_0, which determines σ_{K_1} uniquely, and the claim is proven.

Now let $v \in V_1$, and choose a compact open subgroup K_1 such that $K_1 \subset K_0$ and $v \in V^{K_1}$, and we define $\sigma : V_1 \to V_2$ by $\sigma(v) = \sigma_{K_1}(v)$. We observe that this definition of $\sigma(v)$ does not depend on the choice of K_1. If K_2 is another compact open subgroup such that $K_2 \subset K_0$ and $v \in V^{K_2}$, then $K_1 \cap K_2 \subset K_0$ and $v \in V^{V_1 \cap V_2}$ also. By the previous extension argument, we must have that $\sigma_{K_1 \cap K_2}$ extends σ_0, σ_{K_1}, and σ_{K_2} uniquely, so that $\sigma_{K_1}(v) = \sigma_{K_1 \cap K_2}(v) = \sigma_{K_2}(v)$.

Finally, we show that σ is an intertwining map, so that it gives us the desired isomorphism to conclude that $\pi_1 \cong \pi_2$. Given $v \in V_1$, find a compact open subgroup K_1 of G such that $v \in V_1^{K_1}$ and $\sigma(v) \in V_2^{K_1}$. As in the proof of Proposition 2.4(2), if $\epsilon_{gK_1} = \mu(K_1)^{-1}\mathbf{1}_{gK_1}$, then we have $\pi_1(\epsilon_{gK_1})v = \pi_1(g)v$ and $\pi_2(\epsilon_{gK_1})\sigma(v) = \pi_2(g)\sigma(v)$. Now, we have

$$\sigma(\pi_1(g)v) = \sigma(\epsilon_{gK_1} \cdot v) = \epsilon_{gK_1} \cdot \sigma(v) = \pi_2(g)\sigma(v),$$

and so $\sigma \circ \pi_1(g) = \pi_2(g) \circ \sigma$ for every $g \in G$, and $\pi_1 \cong \pi_2$. $\square$

3. Characters

Let V be a finite dimensional vector space over $\mathbb{C}$, and let $T \in \mathrm{End}(V)$. Let $W \subset V$ be any subspace of V such that $T(V) \subset W$. Then, T restricted to W is an endomorphism of W. Let $w_1, w_2, \ldots, w_n$ be a basis of V such that $w_1, \ldots, w_d$ is a basis of W. Then, if the matrix for $T|_W$ with respect to the basis $w_1, \ldots, w_d$ is the d-by-d matrix A, then the matrix for T with respect to $w_1, \ldots, w_n$ is an n-by-n matrix of the form $\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$, where B is a d-by-$(n-d)$ matrix. In particular, we have $\mathrm{tr}(T) = \mathrm{tr}(T|_W) = \mathrm{tr}(A)$.

Now let V be a vector space which is possibly infinite dimensional over $\mathbb{C}$. Let $T \in \mathrm{End}(V)$ be such that $\dim_{\mathbb{C}}(T(V))$ is finite, or T has *finite rank.* If W is a finite dimensional subspace of V such that $T(V) \subset W$, we define the trace of T by $\mathrm{tr}(T) = \mathrm{tr}(T|_W)$. This definition is independent of the choice of W, since if W' is another finite dimensional subspace of V such that $T(V) \subset W'$, then $\mathrm{tr}(T|_W) = \mathrm{tr}(T|_{W'})$ by the previous discussion.

Let G be a unimodular locally compact totally disconnected group, and let (π, V) be an admissible representation of G. For any $f \in \mathcal{H}(G)$, by Proposition 2.2 there is a compact open subgroup K_0 of G such that $f \in \mathcal{H}_{K_0}(G)$. Now, $\pi(f)$ is an endomorphism of V, and since $\mathcal{H}_{K_0}(G) = \epsilon_{K_0}\mathcal{H}(G)\epsilon_{K_0}$ by Proposition 2.1(2), and $f \in \mathcal{H}_{K_0}(G)$, then $\pi(f)V = f \cdot V \subset \epsilon_{K_0} \cdot V = V^{K_0}$. Since (π, V) is assumed to be admissible, then V^{K_0} is finite dimensional, and so $\pi(f) \in \mathrm{End}(V)$ has finite rank. We define the *character* of the admissible representation (π, V) to be the function $\chi_\pi : \mathcal{H}(G) \to \mathbb{C}$, where

$\chi_\pi(f) = \mathrm{tr}(\pi(f))$. In particular, the character of the admissible representation (π, V) is a *distribution*, which is a linear functional from $C_c^\infty(G)$ to $\mathbb{C}$.

We have the following general algebraic fact which we will apply to our situation.

Proposition 3.1. *Let $\mathcal{A}$ be an algebra over a field Ω, and let V_1 and V_2 be simple $\mathcal{A}$-modules which are finite dimensional as vector spaces over Ω. For any $f \in \mathcal{A}$, multiplication by f induces endomorphisms $\pi_i(f)$ on V_i, for $i = 1, 2$. If $\mathrm{tr}(\pi_1(f)) = \mathrm{tr}(\pi_2(f))$ for all $f \in \mathcal{A}$, then $V_1 \cong V_2$ as $\mathcal{A}$-modules.*

Proof. It follows from the Jacobson density theorem that if $V_1 \ncong V_2$, then there exist elements $p_1, p_2 \in \mathcal{A}$ such that p_i acts as the identity on V_i for $i = 1, 2$, and p_i acts as zero on V_j for $i \neq j$ (see [L, Theorems XVII.3.2 and XVII.3.7]). So, if $V_1 \ncong V_2$, then at least one is nonzero, say V_1, and we have $\mathrm{tr}(\pi_1(p_1)) = \dim_\Omega(V_1) \neq 0$, while $\mathrm{tr}(\pi_2(p_1)) = 0$, contradicting the assumption that these traces are equal. $\square$

In the case of finite dimensional continuous representations of compact groups, we saw that if the characters of two irreducible representations agree, then the representations must be isomorphic. We are now able to obtain the analogous result for admissible representations.

Theorem 3.1. *Let G be a unimodular locally compact totally disconnected group, and let (π_1, V_1) and (π_2, V_2) be irreducible admissible representations of G with characters χ_1 and χ_2, respectively. If $\chi_1(f) = \chi_2(f)$ for every $f \in \mathcal{H}(G)$, then $\pi_1 \cong \pi_2$.*

Proof. Let K_0 be any compact open subgroup of G, and consider $V_1^{K_0}$ and $V_2^{K_0}$ as $\mathcal{H}_{K_0}(G)$-modules, which are finite dimensional as vector spaces over $\mathbb{C}$, since (π_1, V_1) and (π_2, V_2) are admissible. For any $\phi \in \mathcal{H}_{K_0}(G)$, we are assuming that $\mathrm{tr}(\pi_1(\phi)) = \mathrm{tr}(\pi_2(\phi))$, and it follows from Proposition 3.1, with $\Omega = \mathbb{C}$, that $V_1^{K_0} \cong V_2^{K_0}$ as $\mathcal{H}_{K_0}(G)$-modules. Applying Proposition 2.7, we have $\pi_1 \cong \pi_2$. $\square$

4. The contragredient of a smooth representation

Let Γ be a compact group, and (π, V) a continuous finite dimensional representation of Γ. Recall that the contragredient representation of (π, V), denoted $(\hat{\pi}, V^*)$, is a representation of Γ acting on the dual space V^* of V, where if $g \in \Gamma$, $v \in V$, and $L \in V^*$, the action is defined by $(\hat{\pi}(g)L)(v) = L(\pi(g^{-1})v)$. We may also consider the double contragredient, $(\hat{\hat{\pi}}, V^{**})$, which then acts on the double dual V^{**} by $(\hat{\hat{\pi}}(g)\Lambda)(L) = \Lambda(\hat{\pi}(g^{-1})L)$, for $\Lambda \in V^{**}$, $L \in V^*$, and $g \in \Gamma$. We have a canonical isomorphism of vector spaces $T : V \to V^{**}$, defined by $(T(v))(L) = L(v)$, for $v \in V$ and $L \in V^*$. If $g \in \Gamma$, then we have

$$\begin{aligned}\big((T \circ \pi(g))(v)\big)(L) &= T(\pi(g)v)(L) = L(\pi(g)v) \\ &= (\hat{\pi}(g^{-1})L)(v) = T(v)(\hat{\pi}(g^{-1})L) \\ &= (\hat{\hat{\pi}}(g)T(v))(L) = \big((\hat{\hat{\pi}}(g) \circ T)(v)\big)(L).\end{aligned}$$

That is, T is an intertwining operator, so we have $\hat{\hat{\pi}} \cong \pi$.

Now let G be a locally compact totally disconnected group, and let (π, V) be a smooth representation of G. A linear functional $L \in V^*$ is called *π-smooth* if there is an open subgroup H of G such that $L(\pi(h)v) = L(v)$ for every $h \in H$ and $v \in V$. We define the *π-smooth dual* of V, denoted $\hat{V}$, to be the space of all π-smooth linear functionals in V^*, which is a subspace of V^*.

Lemma 4.1. *Let G be a locally compact totally disconnected group, and (π, V) a smooth representation of G. If V is finite dimensional, then $V^* = \hat{V}$.*

Proof. Let $v_1, \ldots, v_n$ be a basis for V. Since (π, V) is smooth, then $\mathrm{stab}_G(v_i)$ is an open subgroup of G for each i. Let $H = \cap_{i=1}^n \mathrm{stab}_G(v_i)$, which is an open subgroup of G. For any $L \in V^*$, $h \in H$, $v \in V$, we have $L(\pi(h)v) = L(v)$, and so L is π-smooth. □

If (π, V) is any smooth representation of G, for $g \in G$, $L \in \hat{V}$, and $v \in V$, define $\hat{\pi}(g)L \in V^*$ by $(\hat{\pi}(g)L)(v) = L(\pi(g^{-1})v)$.

Proposition 4.1. *Let (π, V) be a smooth representation of G. If $L \in \hat{V}$ and $g \in G$, then $\hat{\pi}(g)L \in \hat{V}$, and $(\hat{\pi}, \hat{V})$ is a smooth representation of G.*

Proof. Since $L \in \hat{V}$, there is an open subgroup H of G such that $L(\pi(h)v) = L(v)$ for all $h \in H$ and $v \in V$. In particular, if $g \in G$ and $v \in V$, then $L(\pi(h)\pi(g^{-1})v) = L(\pi(g^{-1})v)$. If $g \in G$, let $H_1 = gHg^{-1}$, which is also an open subgroup of G. Let $h_1 \in H_1$, where $h_1 = ghg^{-1}$ for some $h \in H$. Then we have, for any $v \in V$,

$$(\hat{\pi}(g)L)(\pi(h_1)v) = L(\pi(hg^{-1})v) = L(\pi(g^{-1})v) = (\hat{\pi}(g)L)(v).$$

Thus, $\hat{\pi}(g)L \in \hat{V}$.

This gives a homomorphism $\hat{\pi} : G \to \mathrm{Aut}(\hat{V})$, which follows for the exact same reason as in the case of a compact group. To check that $(\hat{\pi}, \hat{V})$ is smooth, let $L \in \hat{V}$, and we have

$$\mathrm{stab}_G(L) = \{g \in G \mid L(\pi(g^{-1})v) = L(v) \text{ for every } v \in V\}.$$

Since L is π-smooth, there is an open subgroup H of G such that $H \subset \mathrm{stab}_G(L)$, and thus the stabilizer of L is open and $(\hat{\pi}, \hat{V})$ is a smooth representation of G. □

The representation $(\hat{\pi}, \hat{V})$ is called the *smooth contragredient* of (π, V) (or just the *contragredient* if smoothness is clear from the context).

Let I be any set, let $\{V_\alpha\}_{\alpha \in I}$ be a collection of $\mathbb{C}$-vector spaces which are indexed by I, and consider the vector space $V = \oplus_{\alpha \in I} V_\alpha$. Elements of V consists of vectors of the form (v_α), where $v_\alpha \in V_\alpha$, and $v_\alpha = 0$ for all but finitely many indices $\alpha \in I$. An element $L \in V^*$ may be then be written as (L_α), where $L_\alpha \in V_\alpha^*$, but there is no restriction as to the number of nonzero terms. For this $L \in V^*$, $v \in V$, we have $L(v) = \sum_{\alpha \in I} L_\alpha(v_\alpha)$, where the sum is actually finite since all but finitely many of the v_α are 0. In particular, we have $V^* = \prod_{\alpha \in I} V_\alpha^*$.

In the case that (π, V) is a smooth representation of G, and K is a compact open subgroup of G, we have from Theorem 1.1 that $V = \oplus_{\rho \in \mathrm{Irr}(K)} V(\rho)$, where $V(\rho)$ is the ρ-isotypic part of the representation $(\pi|_K, V)$ of K. Then $V^* = \prod_{\rho \in \mathrm{Irr}(K)} V(\rho)^*$ from the above discussion. In the case that (π, V) is an admissible, each $V(\rho)$ is finite dimensional, and the π-smooth dual of V is given as follows.

Lemma 4.2. *Let (π, V) be an admissible representation of G, and K a compact open subgroup of G. Then the π-smooth dual of V is given by*

$$\hat{V} = \bigoplus_{\rho \in \mathrm{Irr}(K)} V(\rho)^*,$$

and for every $\rho \in \mathrm{Irr}(K)$, we have $V(\rho)^* = \hat{V}(\hat{\rho})$.

Proof. For the first statement, we have seen that $V^* = \prod_{\rho\in\mathrm{Irr}(K)} V(\rho)^*$. Let $L = (L_\rho)$, where $L \in \oplus_{\rho\in\mathrm{Irr}(K)} V(\rho)^*$, and say $L_{\rho_i} \in V(\rho_i)^*$ are nonzero for some $i = 1,\dots,k$, and $L_\rho \in V(\rho)$ is zero for any $\rho \neq \rho_i$. Let $K_0 = \cap_{i=1}^k \ker(\rho_i)$, which is an open subgroup of G, and note that $\pi|_K$ acting on $V(\rho)$ has kernel $\ker(\rho)$. For any $v = (v_\rho) \in V$, where $v_\rho \in V(\rho)$, and any $k \in K_0$, we have

$$L(\pi(k)v) = \sum_{i=1}^k L_{\rho_i}(\pi(k)v_{\rho_i}) = \sum_{i=1}^k L_{\rho_i}(v_{\rho_i}) = L(v).$$

So, $L \in \hat{V}$, and $\oplus_{\rho\in\mathrm{Irr}(K)} V(\rho)^* \subset \hat{V}$.

For the second statement, we have that $V(\rho)$ is a finite dimensional representation, with $\pi|_K$ acting on this space as a continuous representation of K, and $V(\rho) = \oplus_{i=1}^k W_i$ for some spaces W_i, where $\pi|_K$ acting on W_i is isomorphic to ρ as a representation of K. Then $V(\rho)^* = \oplus_{i=1}^k W_i^*$, and the contragredient of $\pi|_K$ on W_i^* is isomorphic to $\hat{\rho}$. Moreover, these are the only subspaces of $\hat{V}$ on which the contragredient of $\pi|_K$ acting is isomorphic to $\hat{\rho}$, since we began with the only subspaces of V on which $\pi|_K$ acting is isomorphic to ρ. Hence, $V(\rho)^* = \hat{V}(\hat{\rho})$. □

Theorem 4.1. *Let G be a locally compact totally disconnected group, and (π, V) an admissible representation of G. Then:*

(1) *$(\hat{\pi}, \hat{V})$ is an admissible representation of G, and*
(2) *$\pi \cong \hat{\hat{\pi}}$.*

Proof. (1): Let K be a compact open subgroup of G. For any $\rho \in \mathrm{Irr}(K)$, Lemma 4.2 gives us $V(\rho)^* = \hat{V}(\hat{\rho})$, where $V(\rho)^*$ is finite dimensional, since $V(\rho)$ is finite dimensional by admissibility and Theorem 1.1. Any continuous finite dimensional representation of K is irreducible if and only if its contragredient is irreducible, which is obtained by Schur orthogonality since their characters are complex conjugates. So, for any $\rho \in \mathrm{Irr}(K)$, we have $V(\hat{\rho})^* = \hat{V}(\rho)$ is finite dimensional, and $(\hat{\pi}, \hat{V})$ is admissible by Theorem 1.1.

(2): Let K be a compact open subgroup of G. From Lemma 4.2, we have, for any $\rho \in \mathrm{Irr}(K)$, $V(\rho)^* = \hat{V}(\hat{\rho})$. Since $(\hat{\pi}, \hat{V})$ is admissible from (1), we have $\hat{V}(\hat{\rho})^* = \hat{\hat{V}}(\hat{\hat{\rho}})$. We have observed that $\hat{\hat{\rho}} \cong \rho$, and so $V(\rho)^{**} = \hat{\hat{V}}(\rho)$. Now, we have

$$\hat{\hat{V}} = \bigoplus_{\rho\in\mathrm{Irr}(K)} \hat{\hat{V}}(\rho) = \bigoplus_{\rho\in\mathrm{Irr}(K)} V(\rho)^{**},$$

and since $V(\rho)$ is finite dimensional by admissibility, then $V(\rho)$ is isomorphic to $V(\rho)^{**}$ as vector spaces, with the canonical isomorphism T_ρ, defined by $(T_\rho(v_\rho))(L_\rho) = L_\rho(v_\rho)$, where $v_\rho \in V(\rho)$, $L_\rho \in V(\rho)^*$. By defining $T = \oplus_\rho T_\rho : V \to \hat{\hat{V}}$, we get an isomorphism of vector spaces given by $(T(v))(L) = \sum_\rho L_\rho(v_\rho)$, where $v = (v_\rho) \in \oplus_\rho V(\rho)$ and $L = (L_\rho) \in \oplus_\rho V(\rho)^*$, which is just the canonical map of vector spaces given by $(T(v))(L) = L(v)$. By the exact same calculation as was done previously for compact groups, T is an intertwining operator, and $\pi \cong \hat{\hat{\pi}}$. □

We now consider the case of a unimodular locally compact totally disconnected group, so that we may study the Hecke algebra and the character of an admissible representation in relation to the smooth contragredient. For any $f \in \mathcal{H}(G)$, define f^- by $f^-(g) = f(g^{-1})$, so that $f^- \in \mathcal{H}(G)$.

Lemma 4.3. *Let G be a unimodular locally compact totally disconnected group, let (π, V) be an admissible representation of G, and let $f \in \mathcal{H}(G)$. Then for any $v \in V$ and $L \in \hat{V}$, we have $(\hat{\pi}(f)L)(v) = L(\pi(f^-)v)$.*

Proof. By definition, we have

$$(\hat{\pi}(f)L)(v) = \left(\int_G f(g)\hat{\pi}(g)L\, dg\right)(v).$$

Since this integral is just a finite sum, we may bring the v inside of the integral, and we have

$$(\hat{\pi}(f)L)(v) = \int_G f(g)(\hat{\pi}(g)L)(v)\, dg = \int_G f(g)L(\pi(g^{-1})v)\, dg.$$

Again, since the integral is a finite sum, the L may be brought outside of the integral. Also, since G is unimodular, the change of variables $g \mapsto g^{-1}$ does not change the integral. So, we have

$$(\hat{\pi}(f)L)(v) = L\left(\int_G f(g)\pi(g^{-1})v\, dg\right) = L\left(\int_G f(g^{-1})\pi(g)v\, dg\right) = L(\pi(f^-)v). \quad \square$$

Let (π, V) be an admissible representation, and let $[\cdot,\cdot] : V \times \hat{V} \to \mathbb{C}$ be the bilinear form given by the natural pairing $[v, L] = L(v)$, for $v \in V$, $L \in \hat{V}$. It follows from Lemma 4.2 that this pairing is nondegenerate. If K is a compact open subgroup of G, let $[\cdot,\cdot]_K$ denote the pairing $[\cdot,\cdot]$ restricted to $V^K \times \hat{V}^K$. We see now that this is also nondegenerate.

Lemma 4.4. *Let G be a unimodular locally compact totally disconnected group, K a compact open subgroup of G, and (π, V) an admissible representation of G. Then the pairing $[\cdot,\cdot]_K : V^K \times \hat{V}^K \to \mathbb{C}$ is a nondegenerate bilinear map.*

Proof. Let $v \in V^K$ be nonzero. Since the pairing $[\cdot,\cdot]$ on $V \times \hat{V}$ is nondegenerate, we can find an $L_1 \in \hat{V}$ such that $[v, L_1] \neq 0$. Since $v \in V^K = \epsilon_K \cdot V$, we have $\pi(\epsilon_K)v = v$. We have $\epsilon_K^- = \epsilon_K$, and so by Lemma 4.3 we have

$$[v, L_1] = [\pi(\epsilon_K)v, L_1] = [v, \hat{\pi}(\epsilon_K)L_1].$$

Now, $L = \epsilon_K \cdot L_1 \in \epsilon_K \cdot \hat{V} = \hat{V}^K$, and we have found an $L \in \hat{V}^K$ such that $[v, L]_K \neq 0$, and $[\cdot,\cdot]_K$ is nondegenerate in the right variable. If $L \in \hat{V}^K$ is nonzero, there is a $v_1 \in V$ such that $[v_1, L] \neq 0$, and we may apply the exact same argument to see that if $v = \pi(\epsilon_K)v_1$, then $v \in V^K$ and $[v, L]_K \neq 0$. Thus $[\cdot,\cdot]_K$ is a nondegenerate bilinear map. $\square$

Finally, we may relate the character an admissible representation to the character of its contragredient. The following result should be thought of as analogous to the result that the characters of a finite dimensional continuous representation of a compact group and its contragredient are complex conjugates.

Proposition 4.2. *Let G be a unimodular locally compact totally disonnected group, and let (π, V) be an admissible representation of G. For any $f \in \mathcal{H}(G)$, we have $\operatorname{tr}(\hat{\pi}(f)) = \operatorname{tr}(\pi(f^-))$.*

Proof. Given $f \in \mathcal{H}(G)$, choose a compact open subgroup K_0 of G such that $f \in \mathcal{H}_{K_0}(G)$, and notice that we also have $f^- \in \mathcal{H}_{K_0}(G)$. Consider the pairing $[\cdot,\cdot]_{K_0} : V^{K_0} \times \hat{V}^{K_0} \to \mathbb{C}$, which by Lemma 4.4, is a nondegenerate bilinear map. Since (π, V) is admissible, V^{K_0} is finite dimensional, and by Theorem 4.1(1), $(\hat{\pi}, \hat{V})$ is admissible, and so $\hat{V}^{K_0}$ is also finite dimensional. Since $[\cdot,\cdot]_{K_0}$ provides a nondegenerate bilinear map on $V^{K_0} \times \hat{V}^{K_0}$, then in fact we have $\hat{V}^{K_0}$ and $(V^{K_0})^*$ are isomorphic as vector spaces (by [L, Theorem III.6.4], for example). By Lemma 4.3, we have, for any $v \in V^{K_0}$, $L \in \hat{V}^{K_0}$, $[v, \hat{\pi}(f)L]_{K_0} = [\pi(f^-)v, L]_{K_0}$. By choosing a basis for V^{K_0} and a dual basis for $\hat{V}^{K_0}$, the matrices for $\pi(f^-)$ and $\hat{\pi}(f)$ with respect to these respective bases may be seen to be the transpose of each other (since they are adjoint operators with respect to the bilinear map $[\cdot,\cdot]_{K_0}$). Thus, we have $\mathrm{tr}(\hat{\pi}(f)) = \mathrm{tr}(\pi(f^-))$. □

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