THE HAAR MEASURE

MATH 519

1. Measure theory background

In this section, we give a brief review of the measure theory which will be used in later sections. We use [R, Chapters 1 and 2] as our main resource.

A *σ-algebra* on a set X is a collection $\mathcal{M}$ of subsets of X such that $\varnothing \in \mathcal{M}$, if $S \in \mathcal{M}$, then $X \setminus S \in \mathcal{M}$, and if a countable collection $S_1, S_2, \ldots \in \mathcal{M}$, then $\cup_{i=1}^{\infty} S_i \in \mathcal{M}$. That is, $\mathcal{M}$ is closed under complements and countable unions, and contains the empty set. A *measure* on a set X with σ-algebra $\mathcal{M}$ is a function $\mu : \mathcal{M} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that, if $\{S_i\}_{i \geq 1}$ is a countable collection of pairwise dijoint elements of $\mathcal{M}$, then

$$\mu\left(\cup_{i=1}^{\infty} S_i\right) = \sum_{i=1}^{\infty} \mu(S_i),$$

where we define $a + \infty = \infty$ for any $a \in \mathbb{R}_{\geq 0} \cup \{\infty\}$. In this case, the triple $(X, \mathcal{M}, \mu)$ is called a *measure space*, and the elements of $\mathcal{M}$ are called *measurable sets*. We will assume that measures are not trivial, that is, they take a nonzero value on some measurable set.

If X is a topological space, then the smallest σ-algebra containing all of the open sets of X is called the *Borel* σ-algebra, which we denote by $\mathcal{B}$. The elements of $\mathcal{B}$ are called *Borel sets*, and a measure defined on $\mathcal{B}$ is called a *Borel measure*.

Consider $\mathbb{R}$ with the standard metric topology, $\mathbb{R}_{\geq 0}$ with the subspace topology, and consider the collection of Borel sets of $\mathbb{R}_{\geq 0}$. If $(X, \mathcal{M}, \mu)$ is a measure space, a function $f : X \to \mathbb{R}_{\geq 0}$ is called *measurable* if for every Borel set S in $\mathbb{R}_{\geq 0}$, $f^{-1}(S) \in \mathcal{M}$. That is, f is measurable if the inverse image of any Borel set is a measurable set. In the case that X is a topological space and $\mathcal{M}$ is the Borel σ-algebra on X, note that any continuous function $f : X \to \mathbb{R}_{\geq 0}$ is also measurable. In general, if $(Y, \mathcal{N}, \nu)$ is any other measure space, then we call $f : X \to Y$ measurable if $A \in \mathcal{N}$ implies $f^{-1}(A) \in \mathcal{M}$.

Let $(X, \mathcal{M}, \mu)$ be a measure space, and $S \in \mathcal{M}$. Define $\mathbf{1}_S$ to be the characteristic function on S, so $\mathbf{1}_S(x) = 1$ when $x \in S$ and $\mathbf{1}_S(x) = 0$ if $x \notin S$. A *simple function* on X is a function $h : X \to \mathbb{R}_{\geq 0}$ which can be written as

$$h(x) = \sum_{i=1}^{m} \alpha_i \mathbf{1}_{S_i}(x), \qquad \text{for distinct } \alpha_i \in \mathbb{R}_{>0}, \text{ pairwise disjoint } S_i \in \mathcal{M}.$$

That is, a simple function is a non-negative measurable function which takes a finite number of positive real values. For any $A \in \mathcal{M}$ and any simple function $h = \sum_{i=1}^{m} \alpha_i \mathbf{1}_{S_i}$ on X, we define

$$\int_A h \, d\mu = \sum_{i=1}^{m} \alpha_i \mu(A \cap S_i).$$

Now, if $f : X \to \mathbb{R}_{\geq 0}$ is any measurable function, and $A \in \mathcal{M}$, we define the *Lebesgue integral* on f over A as

$$\int_A f \, d\mu = \sup\left\{ \int_A h \, d\mu \;\mid\; h \text{ simple}, h(x) \leq f(x) \text{ for all } x \in X \right\}.$$

We will also use the notation $d\mu(x)$ or simply dx in place of $d\mu$.

For any $\mathbb{R}$-valued function h on X, define $h^+(x) = \max\{h(x), 0\}$ to be the positive part of h, and $h^-(x) = -\min\{h(x), 0\}$ to be the negative part of h, so that $h = h^+ - h^-$. If $f : X \to \mathbb{C}$ is any $\mathbb{C}$-valued measurable function, where $\mathbb{C}$ has the σ-algebra with respect to the standard topology, then write $f = u + iv$, where u and v are both $\mathbb{R}$-valued functions, which are also measurable (Exercise). Assume further that $\int_X |f| \, d\mu < \infty$, that is, $f \in L^1(\mu)$, where $|f|$ is also a measurable function (Exercise). Now we define the integral of the function $f : X \to \mathbb{C}$ over a measurable set A by

$$\int_A f \, d\mu = \int_A u^+ \, d\mu - \int_A u^- \, d\mu + i \int_A v^+ \, d\mu - i \int_A v^- \, d\mu.$$

Note that each of the integrals above on the right is finite, since $u^+ \leq |u| \leq |f|$, and similarly for u^-, v^+, and v^-.

Now let X be a locally compact Hausdorff space, $\mathcal{B}$ the Borel σ-algebra on X, and μ a Borel measure on X. If $A \in \mathcal{B}$, then μ is *outer regular* on A if

$$\mu(A) = \inf\{\mu(U) \;\mid\; A \subset U, U \text{ open}\},$$

and μ is *inner regular* on A if

$$\mu(A) = \sup\{\mu(K) \;\mid\; K \subset A, K \text{ compact}\}.$$

A *regular Borel measure* on the Borel σ-algebra of a locally compact Hausdorff space X is a measure which is outer regular on all Borel sets and inner regular on all open sets.

Let $C_c(X)$ denote the set of all continuous complex-valued functions with compact support on a topological space X. Then $C_c(X)$ is a complex vector space, and a *linear functional* on $C_c(X)$ is just a linear transformation $\Lambda : C_c(X) \to \mathbb{C}$. Let $C_c^+(X)$ denote the set of continuous real-valued functions in $C_c(X)$ which are non-negative, and not the constant zero function. A *positive linear functional* on $C_c(X)$ is a linear functional $\Lambda : C_c(X) \to \mathbb{C}$ such that, when $f \in C_c^+(X)$, then $\Lambda(f) \in \mathbb{R}$ and $\Lambda(f) \geq 0$.

Theorem 1.1 (Riesz Representation Theorem)**.** *Let X be a locally compact Hausdorff space, and let Λ be a positive linear functional on $C_c(X)$. Then there exists a unique regular Borel measure μ on X which is finite on all compact subsets of X, such that*

$$\Lambda(f) = \int_X f \, d\mu \quad \textit{for every } f \in C_c(X).$$

We will also need the following result, a proof of which can be found in [K].

Theorem 1.2 (Urysohn's Lemma)**.** *Let X be a locally compact Hausdorff space, let K be a nonempty compact subset of X, and U an open subset of X such that $K \subset U$. Then there is a function $f \in C_c^+(X)$ such that $f(x) = 1$ when $x \in K$,* $\mathrm{supp}(f) \subset U$, *and $0 \leq f(x) \leq 1$ for all $x \in X$.*

2. Definition and examples of Haar measure

Let G be a locally compact group, and let μ be a Borel measure on G, with $\mathcal{B}$ the Borel σ-algebra of G. The measure μ is *left translation invariant* if for every $S \in \mathcal{B}$, $\mu(gS) = \mu(S)$ for all $g \in G$, and is *right invariant* if for every $S \in \mathcal{B}$, $\mu(Sg) = \mu(S)$ for all $g \in G$.

A *left* (respectively, *right*) *Haar measure* on G is a regular Borel measure on G which is finite on compact subsets and left (respectively, right) translation invariant. The most familiar example of a Haar measure is the Lebesgue measure on $\mathbb{R}^n$, viewed as an additive group. In this case, since the group is abelian, the Haar measure is both right and left translation invariant, or is *bi-invariant*.

Recall that for $g \in G$ and a function $f : G \to \mathbb{C}$, we define $L_g f$ by $L_g f(x) = f(g^{-1}x)$, and $R_g f$ by $R_g f(x) = f(xg)$.

Proposition 2.1. *Let G be a locally compact group, and let μ be a regular Borel measure on G which is finite on all compact subsets of G.*

(1) *The measure μ is a left Haar measure on G if and only if the measure $\tilde{\mu}$, defined by $\tilde{\mu}(A) = \mu(A^{-1})$ for $A \in \mathcal{B}$, is a right Haar measure on G.*
(2) *If μ is a left Haar measure on G, and ϕ is a continuous automorphism of G with continuous inverse, then $\mu \circ \phi$ is a left Haar measure on G.*
(3) *The measure μ is a left Haar measure on G if and only if for every function $f \in C_c^+(G)$,*
$$\int_G L_g f \, d\mu = \int_G f \, d\mu \quad \textit{for all } g \in G.$$
(4) *If μ is a left Haar measure on G, then μ is positive on all nonempty open subsets of G, and*
$$\int_G f \, d\mu > 0 \quad \textit{for all } f \in C_c^+(G).$$
(5) *If μ is a left Haar measure on G, then $\mu(G)$ is finite if and only if G is compact.*

Proof. (1): We know that $U \subset G$ is open if and only if U^{-1} is open in G. Consider the collection of subsets S of G such that S^{-1} is Borel, which is a σ-algebra (Exercise). Since this σ-algebra contains all open sets, it contains all Borel sets, from which it follows that a subset A of G is a Borel set if and only if A^{-1} is. Since the inverse map is continuous, K is compact if and only K^{-1} is compact. It follows directly from these facts that $\tilde{\mu}$ is a Borel measure which is finite on compact subsets, since μ is. For outer regularity, let A be a Borel set. Then $\tilde{\mu}(A) = \mu(A^{-1})$, where A^{-1} is a Borel set, and so $\tilde{\mu}(A) = \inf\{\mu(U) \mid A^{-1} \subset U, U \text{ open}\}$. Since U is open if and only if U^{-1} is open, and $A^{-1} \subset U^{-1}$ if and only if $A \subset U$, we have $\tilde{\mu}(A) = \inf\{\tilde{\mu}(U) \mid A \subset U, U \text{ open}\}$, and so $\tilde{\mu}$ is outer regular on Borel sets. Inner regularity of $\tilde{\mu}$ follows similarly.

Now notice that for every Borel set A, $\tilde{\mu}(A) = \tilde{\mu}(Ag)$ for every $g \in G$ if and only $\mu(A^{-1}) = \mu(g^{-1}A^{-1})$ for every $g \in G$, if and only if $\mu(A^{-1}) = \mu(gA^{-1})$ for every $g \in G$. It follows that μ is a left Haar measure if and only if $\tilde{\mu}$ is a right Haar measure.

(2): Similar to (1), since ϕ and its inverse are continuous, U is open in G if and only if $\phi(U)$ is open in G, A is a Borel set in G if and only if $\phi(A)$ is a Borel set. and K is compact if and only if $\phi(K)$ is compact. So, following the proof of (1), it follows that $\mu \circ \phi$ is a regular Borel measure which is finite on compact subsets. Now let A

be any Borel set, and $g \in G$. Then we have, since ϕ is an automorphism and μ is a left Haar measure, $(\mu\circ\phi)(gA) = \mu(\phi(g)\phi(A)) = (\mu\circ\phi)(A)$, and so $\mu\circ\phi$ is a left Haar measure.

(3): First suppose that μ is a left Haar measure on G. Then for any simple function h, we have

$$\int_G L_g h \, d\mu = \int_G h \, d\mu \quad \text{for any } g \in G.$$

Now if $f : G \to \mathbb{R}_{\geq 0}$ is a measurable function, then by definition, for any $g \in G$,

$$\int_G L_g f \, d\mu = \sup\left\{ \int_G h \, d\mu \;\mid\; h \text{ simple}, \, h \leq L_g f \right\}.$$

Since the integral is left invariant on simple functions, and $h \leq L_g f$ if and only $L_{g^{-1}} h \leq f$, we have

$$\int_G L_g f \, d\mu = \sup\left\{ \int_G L_{g^{-1}} h \, d\mu \;\mid\; h \text{ simple}, \, L_{g^{-1}} h \leq f \right\}.$$

Finally, h is simple if and only if $L_g h$ is simple, and so replacing h by $L_g h$ gives

$$\int_G L_g f \, d\mu = \sup\left\{ \int_G h \, d\mu \;\mid\; h \text{ simple}, \, h \leq f \right\} = \int_G f \, d\mu,$$

as so we have proven the claim for any measurable $f : G \to \mathbb{R}_{\geq 0}$, and so in particular it holds for $f \in C_c^+(G)$.

Now suppose that for any function $f \in C_c^+(G)$ and any $g \in G$, we have $\int_G L_g f \, d\mu = \int_G f \, d\mu$. Since μ is assumed to be a regular Borel measure, then for every open set U in G, we have

$$\mu(U) = \sup\left\{ \mu(K) \;\mid\; K \subset U, K \text{ compact} \right\}.$$

By Theorem 1.2, for every nonempty compact $K \subset U$, there is a function $f \in C_c^+(G)$ such that $\mathrm{supp}(f) \subset U$, $f(x) = 1$ when $x \in K$, and $||f||_\infty = 1$. For such an f, we have $\mu(K) \leq \int_G f \, d\mu \leq \mu(U)$. It follows that we have

$$\mu(U) = \sup\left\{ \int_G f \, d\mu \;\mid\; f \in C_c^+(G), ||f||_\infty \leq 1, \mathrm{supp}(f) \subset U \right\}.$$

Note that for any $g \in G$, $f \in C_c^+(G)$ and $||f||_\infty \leq 1$ if and only if $L_{g^{-1}} f \in C_c^+(G)$ and $||L_{g^{-1}} f||_\infty \leq 1$. So,

$$\mu(U) = \sup\left\{ \int_G L_{g^{-1}} f \, d\mu \;\mid\; f \in C_c^+(G), ||f||_\infty \leq 1, \mathrm{supp}(L_{g^{-1}} f) \subset U \right\}.$$

The integral is left invariant on $C_c^+(G)$, and we have $\mathrm{supp}(L_{g^{-1}} f) \subset U$ if and only if $\mathrm{supp}(f) \subset gU$, so

$$\mu(U) = \sup\left\{ \int_G f \, d\mu \;\mid\; f \in C_c^+(G), ||f||_\infty \leq 1, \mathrm{supp}(f) \subset gU \right\} = \mu(gU).$$

So, μ is left translation invariant on open sets. If A is any Borel set and $g \in G$, then by outer regularity,

$$\mu(gA) = \inf\{\mu(U) \;\mid\; gA \subset U, U \text{ open}\}.$$

Since U is open if and only if gU is open, and $\mu(gU) = \mu(U)$ when U is open, we obtain $\mu(gA) = \mu(A)$, and so μ is a left Haar measure.

(4): See Problem set 3.

(5): If G is compact, then by definition $\mu(G)$ is finite when μ is a left Haar measure. So suppose that G is not compact, and let K be a compact neighborhood of the identity in G, and let U be a symmetric neighborhood of the identity such that $UU \subset K$. Since G is not compact, no finite number of translates of K can cover G. So, let $\{x_j\}$ be a sequence of points in G so that for every $n \geq 1$, $x_n \notin \cup_{j<n} x_j K$. Suppose that for some pair i, j with $j > i$ we have $x_i U \cap x_j U \neq \varnothing$, so that $x_i u = x_j v$ for some $u, v \in U$. But then $x_j = x_i u v^{-1}$, and since U is symmetric and $UU \subset K$, we have $x_j \in x_i K$, a contradiction. So, the translates $x_i U$, $i \geq 1$, are all pairwise disjoint. Since μ is a left Haar measure, and by (4) above, $\mu(x_i U) = \mu(U) > 0$ for each x_i, and so $\mu(G)$ cannot be finite by countable additivity of the measure μ. □

Parts (2) through (5) of Proposition 2.1 may be stated for right Haar measures, with the same proof as for left Haar measures. From part (3) of Proposition 2.1, we see that to specify a left Haar measure μ on G, it is enough to specify a measure in terms of integration, $\int \, d\mu$, such that $\int_G L_g f \, d\mu = \int_G f \, d\mu$ for every measurable $f : G \to \mathbb{R}_{\geq 0}$, or just every $f \in C_c^+(G)$, and every $g \in G$. Then the measure of a Borel set A is given by $\int_A \, d\mu$. The following is the main result, which will be proven in Sections 3 and 4 below.

Theorem 2.1 (Existence and Uniqueness of Haar measure)**.** *If G is a locally compact group, then there exists a left (and right) Haar measure on G which is unique up to scalar multiple.*

If $x \in G$, then the map $\phi_x : g \mapsto x^{-1} g x$ is a continuous automorphism of G with continuous inverse. If μ is a left Haar measure on G, then by part (2) of Proposition 2.1, $\mu \circ \phi_x$ is also a left Haar measure on G. By Theorem 2.1, there is some positive scalar $\delta_G(x)$ such that for every Borel set A of G, $(\mu \circ \phi_x)(A) = \delta_G(x)\mu(A)$. Note that $\delta_G(x)$ is independent of the initial choice of μ, since μ is unique up to scalar multiple.

Proposition 2.2. *The function $\delta_G : G \to \mathbb{R}_{>0}^{\times}$ is a continuous homomorphism.*

Proof. For any $x_1, x_2 \in G$, we have $\phi_{x_1 x_2} = \phi_{x_2} \circ \phi_{x_1}$, from which it follows that $\delta_G(x_1 x_2) = \delta_G(x_2)\delta_G(x_1) = \delta_G(x_1)\delta_G(x_2)$ in $\mathbb{R}_{>0}^{\times}$, and so δ_G is a multiplicative homomorphism.

Let $z \in \mathbb{R}_{>0}^{\times}$, and let $y < z$ be in the image of δ_G, where $\delta_G(a) = y$, and let $\varepsilon > 0$. Let K be a compact neighborhood of 1 in G, so $\mu(K)$ is finite, and $\mu(K) > 0$ by part (4) of Proposition 2.1, since K contains an open neighborhood of 1. Then $a^{-1}Ka$ is compact, and is in particular a Borel set, since it is closed. Since μ is outer regular on Borel sets, then there exists an open set U, $a^{-1}Ka \subset U$, such that $\mu(U) \leq \mu(a^{-1}Ka) + \mu(K)\varepsilon/2$. Since the map $g \mapsto a^{-1} g a$ is continuous, and $a^{-1}Ka \subset U$, then there is a neighborhood W of a such that $W^{-1}KW \subset U$. So, for any $b \in W$, we have $\mu(b^{-1}Kb) \leq \mu(U)$, so that

$$\mu(b^{-1}Kb) \leq \mu(a^{-1}Ka) + \mu(K)\varepsilon/2.$$

Since $\mu(x^{-1}Kx) = \delta_G(x)\mu(K)$, we have, for every $b \in W$,

$$0 < \delta_G(b) \leq \delta_G(a) + \varepsilon/2 < \delta_G(a) + \varepsilon.$$

In other words, the inverse image $\delta_G^{-1}((0, z))$ of the open interval $(0, z)$ is open in G. Since δ_G is a homomorphism, then $\delta_G(x)^{-1} = \delta_G(x^{-1})$ for any $x \in G$. Now, for any $s \in \mathbb{R}_{>0}$, we may pull back the open interval (s, ∞), and see that its inverse image $\delta_G^{-1}((s, \infty))$ is open in G as well. The inverse image of any open interval $(s, z) \subset \mathbb{R}_{>0}$ is now seen to be open, and so δ_G is continuous. □

Now let μ_L denote the left Haar measure on G. Since δ_G is continuous, the map $f \mapsto \int_G f(g)\, \delta_G(g) d\mu_L(g)$ is a positive linear functional on $C_c(G)$, and so by Theorem 1.1, it corresponds to a regular Borel measure on G which is positive on compact sets.

Proposition 2.3. *Let μ_L be a left Haar measure on G. Then for any measurable function $F : G \to \mathbb{R}_{\geq 0}$, we have $\int_G F \circ \phi_x \, d\mu_L = \delta_G(x) \int_G F \, d\mu_L$. Also, the measure corresponding to the positive linear functional $f \mapsto \int_G f(g)\, \delta_G(g) d\mu_L(g)$ is a right Haar measure on G.*

Proof. From the definition of δ_G, for any $x \in G$ and any simple function h, we have $\int_G h \circ \phi_x \, d\mu_L = \delta_G(x) \int_G h \, d\mu_L$. If $F : G \to \mathbb{R}_{\geq 0}$ is any measurable function, then we have

$$\int_G F \circ \phi_x \, d\mu_L = \sup \left\{ \int_G h \, d\mu_L \;\middle|\; h \text{ simple, } h \leq F \circ \phi_x \right\}.$$

Now, h is simple if and only if $h \circ \phi_x$ is simple, and $h \circ \phi_x \leq F \circ \phi_x$ if and only if $h \leq F$, and so replacing h by $h \circ \phi_x$ yields

$$\int_G F \circ \phi_x \, d\mu_L = \sup \left\{ \int_G h \circ \phi_x \, d\mu_L \;\middle|\; h \text{ simple, } h \leq F \right\} = \delta_G(x) \int_G F \, d\mu_L. \tag{2.1}$$

From the right Haar measure version of part (3) of Proposition 2.1, to prove the second statement we need to show that for any $x \in G$ and any $f \in C_c^+(G)$, we have $\int_G R_x f(g)\, \delta_G(g) d\mu_L(g) = \int_G f(g)\, \delta_G(g) d\mu_L(g)$. For any $x \in G$ and function $f \in C_c^+(G)$, the product $(R_x f)\delta_G$ is in $C_c^+(G)$. Since μ_L is a left Haar measure, then by part (3) of Proposition 2.1, we have

$$\int_G R_x f(g)\, \delta_G(g) d\mu_L(g) = \int_G L_x(R_x f(g) \delta_G(g)) \, d\mu_L(g) = \int_G f(x^{-1}gx) \delta_G(x^{-1}g) \, d\mu_L(g).$$

From Proposition 2.2, δ_G is a homomorphism, and so $\delta_G(x^{-1}g) = \delta_G(x^{-1})\delta_G(g)$, and $\delta(g) = \delta(x^{-1}gx)$. From the above, we have

$$\int_G R_x f(g)\, \delta_G(g) d\mu_L(g) = \delta_G(x^{-1}) \int_G f(x^{-1}gx) \delta_G(x^{-1}gx) \, d\mu_L(g).$$

Applying (2.1), we have

$$\int_G R_x f(g)\, \delta_G(g) d\mu_L(g) = \delta_G(x^{-1})\delta_G(x) \int_G f(g)\, \delta_G(g) d\mu_L(g) = \int_G f(g)\, \delta_G(g) d\mu_L(g),$$

giving right translation invariance of the integral. □

The function δ_G is called the *modular quasicharacter* of the locally compact group G. If μ_L is a left Haar measure for G, then for any $x \in G$ and measurable set A, we have $\mu_L(Ax) = \mu_L(x^{-1}Ax) = \delta_G(x)\mu_L(A)$. That is, μ_L is bi-invariant, and thus a right Haar measure, if and only if $\delta_G(x) = 1$ for every $x \in G$. A locally compact group for which every left Haar measure is also a right Haar measure is called a *unimodular* group. It is immediate that abelian groups are unimodular. We also have the following.

Proposition 2.4. *Every compact group is unimodular.*

Proof. From Proposition 2.2, the function $\delta_G : G \to \mathbb{R}_{>0}^\times$ is a continuous homomorphism. If G is a compact group, then $\delta_G(G)$ is a compact subgroup of $\mathbb{R}_{>0}^\times$, the only possibility of which is $\{1\}$. Thus $\delta_G(x) = 1$ for all $x \in G$, and G is unimodular. □

For some examples of computing the Haar measure of a group, it is useful to remember the change of variables formula for integrating functions on $\mathbb{R}^m$ (see [R, Theorem 7.26]), which is as follows. Let $U \subset \mathbb{R}^m$ be open, $\theta : U \to \mathbb{R}_{\geq 0}$ a measurable function, $T : U \to \mathbb{R}^m$ an injective differentiable function, and let $\int d\mathbf{x} = \int dx_1\, dx_2 \cdots dx_m$ denote the Lebesgue measure on $\mathbb{R}^m$. Then

$$\int_{T(U)} \theta \, d\mathbf{x} = \int_U \theta \circ T \, |J_T| \, d\mathbf{x},$$

where $|J_T|$ is the absolute value of the Jacobian determinant for the function T.

Example 1. ([B2, Chapter 1]) Consider the group $G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \mid x, y \in \mathbb{R}, x > 0 \right\}$, which is homeomorphic to the open set $\mathbb{R}_{>0} \times \mathbb{R}$ in $\mathbb{R}^2$. Let $t = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in G$, and consider left multiplication on G by t. This gives the map $x \mapsto ax$ and $y \mapsto ay + b$ of $\mathbb{R}_{>0} \times \mathbb{R}$ onto itself, and let us call this linear map T. Then $|J_T| = a^2$. Now let $F(x, y) = F : G \to \mathbb{R}_{\geq 0}$ be any Lebesgue measurable function, so that $\theta(x, y) = F(x, y)x^{-2}$ is also Lebesgue measurable. By the change of variables formula, we have

$$\int_G F(x, y)x^{-2} \, dx \, dy = \int_G \theta \circ T \; a^2 \, dx \, dy = \int_G (F \circ T)(x, y) \, x^{-2} \, dx \, dy.$$

In other words, if $g = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$, then for any $t \in G$, we have

$$\int_G F(tg) \, x^{-2} \, dx \, dy = \int_G F(g) \, x^{-2} \, dx \, dy,$$

and thus $\int \; x^{-2} \, dx \, dy$ is the left Haar measure on G.

We can also consider right multiplication on G by t, which gives the linear map $S : x \mapsto ax, y \mapsto bx + y$, where $|J_S| = a$. In this case, the change of variables formula gives us

$$\int_G F(gt) \, x^{-1} \, dx \, dy = \int_G F(g) \, x^{-1} \, dx \, dy,$$

so that $\int \; x^{-1} \, dx \, dy$ is the right Haar measure on G. In particular, G is not unimodular, and by Proposition 2.3 the modular quasicharacter of G is given by $\delta_G(g) = x$.

Example 2. Let F be a non-archimedean local field with ring of integers $\mathcal{O}$, uniformizer π, and $\mathfrak{p} = \pi\mathcal{O}$. Let q be the cardinality of the residue field $\mathcal{O}/\mathfrak{p}$, and let us normalize the absolute value so that $|\pi|_v = q^{-1}$. View F as an additive locally compact group, which is unimodular since it is abelian, and let μ be the Haar measure normalized so that $\mu(\mathcal{O}) = 1$. If $S \subset \mathcal{O}$ is a set of representatives of $\mathcal{O}/\mathfrak{p}$ with $0 \in S$, then $\mathcal{O}$ is the disjoint union of $a + \pi\mathcal{O}$, with $a \in S$. Since μ is the additive Haar measure, we have $\mu(a + \pi\mathcal{O}) = \mu(\pi\mathcal{O})$ for every $a \in S$, and so countable additivity,

$$1 = \mu(\mathcal{O}) = \sum_{a \in S} \mu(a + \pi\mathcal{O}) = q \cdot \mu(\pi\mathcal{O}),$$

and so $\mu(\pi\mathcal{O}) = q^{-1} = |\pi|_v$. We also have $\pi\mathcal{O}$ is the disjoint union of $a\pi + \pi^2\mathcal{O}$, $a \in S$, and we again use countable additivity to obtain $\mu(\pi^2\mathcal{O}) = q^{-2} = |\pi^2|_v$, and we can prove by induction that $\mu(\pi^k\mathcal{O}) = q^{-k} = |\pi^k|_v$ for $k \geq 0$. Similarly, $\pi^{-1}\mathcal{O}$ is the disjoint union of the sets $a\pi^{-1} + \mathcal{O}$, $a \in S$, and we have $\mu(\pi^{-1}\mathcal{O}) = q = |\pi^{-1}|_v$, and in general

$\mu(\pi^m\mathcal{O}) = q^{-m} = |\pi^m|_v$ for any $m \in \mathbb{Z}$. For any $\alpha \in F^\times$, we have $\alpha = \pi^m u$ for some $m \in \mathbb{Z}$, $u \in \mathcal{O}^\times$. Then we have

$$\mu(\alpha\mathcal{O}) = \mu(\pi^m\mathcal{O}) = q^{-m} = |\alpha|_v.$$

Now fix some $c \in F^\times$. Then the map $f_c : x \mapsto cx$ is a continuous automorphism with continuous inverse on the additive group F, and so by part (2) of Proposition 2.1, $\mu \circ f_c$ is a Haar measure on F also. By the uniqueness of Haar measure, we then must have $\mu \circ f_c$ is a scalar multiple of μ. We have already shown that $(\mu \circ f_c)(\mathcal{O}) = \mu(c\mathcal{O}) = |c|_v\mu(\mathcal{O})$, and so we must have $\mu(cA) = |c|_v\mu(A)$ for any Borel set A. This can be used to show that the Haar measure on the multiplicative group $F^\times$ is given by $\int |x|_v^{-1} d\mu(x)$ (see Problem set 3).

Example 3. Consider the additive group $M_n(\mathbb{R})$, which is isomorphic to $\mathbb{R}^{n^2}$ as an additive topological group. So, the Haar measure for $M_n(\mathbb{R})$ is just the Lebesgue measure on $\mathbb{R}^{n^2}$, which we denote by $\int dX = \int dx_{11}\,dx_{12}\cdots dx_{nn}$. Now let $G = GL(n,\mathbb{R})$, which is an open subset of $\mathbb{R}^{n^2}$. Let $t = (t_{ij}) \in G$, and consider left multiplication by t on G. If $X = (x_{ij}) \in G$, then left multiplication by t induces the function $T : x_{ij} \mapsto \sum_{k=1}^n t_{ik}x_{kj}$. When taking the Jacobian of a function from $\mathbb{R}^{n^2}$ to itself, we note that it does not matter which order we take the variables, since permuting variables has the effect of permuting rows, and permuting the corresponding columns in the same way. If we order the variables as $x_{11}, x_{21}, \ldots, x_{n1}, x_{12}, \ldots, x_{n2}, \ldots, x_{nn}$, then the Jacobian J_T is exactly the determinant of the n^2-by-n^2 matrix with n copies of the matrix t down the block diagonal, with 0's elsewhere. That is, $|J_T| = |\det(t)|^n$. Now, if F is a measurable function on G, then $\theta(X) = F(X)|\det(X)|^{-n}$ is also measurable, and from the change of variables formula,

$$\int_G F(X)|\det(X)|^{-n}\,dX = \int_G \theta \circ T\,|\det(t)|^n\,dX = \int_G (F \circ T)(X)\,|\det(X)|^{-n}\,dX.$$

So, we have $\int_G F(tX)\ |\det(X)|^{-n}\,dX = \int_G F(X)\ |\det(X)|^{-n}\,dX$, and the left Haar measure on G is $\int |\det(X)|^{-n}\,dX$.

If we consider right multiplication by t on G, this gives the function $S : x_{ij} \mapsto \sum_{k=1}^n t_{kj}x_{ik}$. When computing the Jacobian J_S, if we order the variables as $x_{11}, x_{12}, \ldots$, $x_{1n}, \ldots, x_{nn}$, then J_S is the determinant of the n^2-by-n^2 matrix with n copies of the transpose of t down the block diagonal, with 0's elsewhere. Since taking the transpose does not change the determinant, we have $|J_S| = |\det(t)|^n$. By applying the change of variables formula again, we obtain that $\int |\det(X)|^{-n}\,dX$ is also a right Haar measure, and $\mathrm{GL}(n,\mathbb{R})$ is unimodular.

Another way to see that this is the Haar measure on $\mathrm{GL}(n,\mathbb{R})$ is to check the invariance under multiplication by each type of elementary matrix. Using this method, one can show that if F is a non-archimedean local field as in Example 2, then the Haar measure of $G = \mathrm{GL}(n,F)$ is $\int |\det(X)|_v^{-n}\,d\mu(X)$, where $\mu(X)$ is the additive Haar measure on $M_n(F)$, or the product of n^2 copies of the additive Haar measure μ on F (see Problem set 3).

3. The Haar covering number

Let G be a locally compact group, and let $f, \varphi \in C_c^+(G)$. Since $\mathrm{supp}(f)$ is compact, it is covered by a finite number of translates of any open set in G. In particular, if $U = \{s \in G \mid \varphi(s) > ||\varphi||_\infty/2\}$, then there are some elements $s_1, \ldots, s_n \in G$ such that

$\operatorname{supp}(f) \subset \cup_{j=1}^n s_j U$. That is, for any $x \in \operatorname{supp}(f)$, there is an s_j such that $s_j^{-1}x \in U$, from which it follows that

$$f \leq \frac{2||f||_\infty}{||\varphi||_\infty} \sum_{j=1}^n L_{s_j}\varphi.$$

We may then define the *Haar covering number of f with respect to φ*, written $(f : \varphi)$, as

$$(f : \varphi) = \inf \left\{ \sum_{j=1}^n c_j \ \mid n \geq 1,\ c_1, \ldots, c_n > 0,\ f \leq \sum_{j=1}^n c_j L_{s_j}\varphi \text{ for some } s_1, \ldots, s_n \in G \right\}.$$

Note that it follows immediately from this definition that for $f_1, f_2 \in C_c^+(G)$,

$$(f_1 + f_2 : \varphi) \leq (f_1 : \varphi) + (f_2 : \varphi), \quad \text{and if } f_1 \leq f_2, \text{ then } (f_1 : \varphi) \leq (f_2 : \varphi). \tag{3.1}$$

The following are other basic properties of the Haar covering number.

Lemma 3.1. *Let G be a locally compact group, and let $f, f_1, f_2, \varphi \in C_c^+(G)$. Then:*

1. *For any $c > 0$, $(cf : \varphi) = c(f : \varphi)$.*
2. *For any $s \in G$, $(f : \varphi) = (L_s f : \varphi)$.*
3. $(f : \varphi) \geq ||f||_\infty / ||\varphi||_\infty$.
4. $(f_1 : \varphi) \leq (f_1 : f_2)(f_2 : \varphi)$.

Proof. (1): We have $cf \leq \sum_{j=1}^n c_j L_{s_j}\varphi$ if and only if $f \leq \sum_{j=1}^n (c_j/c) L_{s_j}\varphi$, from which the result follows.

(2): If $s \in G$, we have $f \leq \sum_{j=1}^n c_j L_{s_j}\varphi$ if and only if $L_s f \leq \sum_{j=1}^n c_j L_{s_j s}\varphi$, from which it follows that $(f : \varphi) = (L_s f : \varphi)$.

(3): If $f \leq \sum_{j=1}^n c_j L_{s_j}\varphi$, then for any $x \in G$,

$$f(x) \leq \sum_{j=1}^n c_j \varphi(s_j^{-1}x) \leq \left(\sum_{j=1}^n c_j \right) ||\varphi||_\infty.$$

This implies $\sum_{j=1}^n c_j \geq ||f||_\infty / ||\varphi||_\infty$, and so $(f : \varphi) \geq ||f||_\infty / ||\varphi||_\infty$.

(4): Suppose that $f_1 \leq \sum_{j=1}^n c_j L_{s_j} f_2$ and $f_2 \leq \sum_{i=1}^m d_i L_{t_i}\varphi$. Then we have

$$f_1 \leq \sum_{j=1}^n c_j L_{s_j} \left(\sum_{i=1}^m d_i L_{t_i}\varphi \right) = \sum_{j=1}^n c_j \sum_{i=1}^m d_i L_{t_i s_j}\varphi = \sum_{i,j} c_j d_i L_{t_i s_j}\varphi.$$

That is, $(f_1 : \varphi) \leq \sum_{i,j} c_j d_i = (\sum_{j=1}^n c_j)(\sum_{i=1}^m d_i)$, and so $(f_1 : \varphi) \leq (f_1 : f_2)(f_2 : \varphi)$. □

Notice that since $f, \varphi \in C_c^+(G)$, then $||f||_\infty, ||\varphi||_\infty \neq 0$, and part (3) of Lemma 3.1 implies we always have $(f : \varphi) > 0$. Part (1) of Lemma 3.1 and (3.1) say that we have "almost linearity". We use the Haar covering number to build an "approximate linear functional" as follows. Fix $f_0 \in C_c^+(G)$, and for $f, \varphi \in C_c^+(G)$, define $I_\varphi(f)$ by

$$I_\varphi(f) = \frac{(f : \varphi)}{(f_0 : \varphi)}.$$

The following result will be applied to build a linear functional, which will give a Haar measure, from I_φ.

Proposition 3.1. *Fix $f_0 \in C_c^+(G)$, and define $I_\varphi(f)$ for $\varphi, f \in C_c^+(G)$ as above. Then*

$$(f_0 : f)^{-1} \leq I_\varphi(f) \leq (f : f_0).$$

If $f_1, f_2 \in C_c^+(G)$, then for every $\varepsilon > 0$, there is a neighborhood V of 1 in G such that, whenever $V \subset \operatorname{supp}(\varphi)$, we have

$$I_\varphi(f_1) + I_\varphi(f_2) \leq I_\varphi(f_1 + f_2) + \varepsilon.$$

Proof. From part (4) of Lemma 3.1, we have

$$(f : \varphi) \leq (f : f_0)(f_0 : \varphi) \text{ and } (f_0 : \varphi) \leq (f_0 : f)(f : \varphi).$$

Dividing the first inequality by $(f_0 : \varphi)$ and the second inequality by $(f : \varphi)$ gives the desired result, $(f_0 : f)^{-1} \leq I_\varphi(f) \leq (f : f_0)$.

Let $\varepsilon > 0$, and let $f_1, f_2 \in C_c^+(G)$. By Theorem 1.2, there is a function $\psi \in C_c^+(G)$ such that $\psi(g) = 1$ when $g \in \operatorname{supp}(f_1 + f_2) = \operatorname{supp}(f_1) \cup \operatorname{supp}(f_2)$. Choose $\delta > 0$ such that

$$\delta \leq \min\left\{(\varepsilon/4)(f_1 + f_2 : f_0)^{-1}, (\sqrt{\varepsilon}/2)(\psi : f_0)^{-1/2}\right\} \tag{3.2}$$

Define the function $h : G \to \mathbb{R}_{\geq 0}$ by $h = f_1 + f_2 + \delta\psi$, so that $h \in C_c^+(G)$. Now define, for $i = 1, 2$, the functions $h_i : G \to \mathbb{R}_{\geq 0}$ as

$$h_i(x) = \begin{cases} f_i(x)/h(x) & \text{if } f_i(x) \neq 0 \\ 0 & \text{if } f_i(x) = 0, \end{cases}$$

Then $h_i \in C_c^+(G)$ and $f_i = h_i h$ for $i = 1, 2$, and $h_1 + h_2 \leq 1$. Recall (from the *Topological Groups* notes) that h_1 and h_2 are right uniformly continuous. So, there are neighborhoods U_1, U_2, of the identity in G, such that $x \in U_i$ implies $|h_i(tx) - h_i(t)| < \delta$ for all $t \in G$. Letting $V = U_1 \cap U_2$, we have $t^{-1}s \in V$ implies $|h_i(s) - h_i(t)| < \delta$, for $i = 1, 2$.

Now let $\varphi \in C_c^+(G)$ such that $\operatorname{supp}(\varphi) \subset V$. Suppose that we have $h \leq \sum_{j=1}^n c_j L_{s_j}\varphi$ for some $c_j \in \mathbb{R}_{>0}$, $s_j \in G$. Then we have $f_i = h_i h \leq \sum_{j=1}^n c_j (L_{s_j}\varphi) h_i$, for $i = 1, 2$. For any $s \in G$, s_j, $i = 1, 2$, we have

$$\varphi(s_j^{-1}s)h_i(s) \leq \varphi(s_j^{-1}s)(h_i(s_j) + \delta),$$

since if $s_j^{-1}s \in \operatorname{supp}(\varphi) \subset V$, this follows from uniform continuity, and otherwise both sides of the inequality are 0. So, for any $s \in G$, $i = 1, 2$, we have

$$f_i(s) \leq \sum_{j=1}^n c_j \varphi(s_j^{-1}s)(h_i(s_j) + \delta),$$

and so $(f_i : \varphi) \leq \sum_{j=1}^n c_j (h_i(s_j) + \delta)$. Since $h_1 + h_2 \leq 1$, we have

$$(f_1 : \varphi) + (f_2 : \varphi) \leq (1 + 2\delta) \sum_{j=1}^n c_j.$$

Since this is true for any $c_j > 0$, $s_j \in G$, such that $h \leq \sum_{j=1}^n c_j L_{s_j}\varphi$, then in fact we have $(f_1 : \varphi) + (f_2 : \varphi) \leq (1 + 2\delta)(h : \varphi)$. Now, by the definition of I_φ, and applying (3.1) and part (1) of Lemma 3.1, then from $h = f_1 + f_2 + \delta\psi$ we have

$$I_\varphi(f_1) + I_\varphi(f_2) \leq (1 + 2\delta) I_\varphi(h) \leq (1 + 2\delta)(I_\varphi(f_1 + f_2) + \delta I_\varphi(\psi)).$$

The right hand side is $I_\varphi(f_1+f_2)+2\delta I_\varphi(f_1+f_2)+2\delta^2 I_\varphi(\psi)$, and by the first part of the Proposition, we know that $I_\varphi(f_1+f_2) \le (f_1+f_2 : f_0)$, and $I_\varphi(\psi) \le (\psi : f_0)$. From our choice of δ in (3.2), we have

$$2\delta I_\varphi(f_1+f_2)+2\delta^2 I_\varphi(\psi) \le 2\delta(f_1+f_2 : f_0)+2\delta^2(\psi : f_0) \le \varepsilon,$$

and so $I_\varphi(f_1)+I_\varphi(f_2) \le I_\varphi(f_1+f_2)+\varepsilon$ whenever $\mathrm{supp}(\varphi) \subset V$, as claimed. $\square$

4. Existence and uniqueness of the Haar measure

In this section, we give a proof of Theorem 2.1, following [RV, Section 1.3] (a similar proof is in [MZ]). For a locally compact group G, we prove the existence and uniqueness up to scalar multiple of a left Haar measure on G, and the result for a right Haar measure follows from part (1) of Proposition 2.1. From Theorem 1.1 and part (3) of Proposition 2.1, to show the existence of a left Haar measure on G, it is enough to show the existence of a positive linear functional $I : C_c(G) \to \mathbb{C}$ such that $I(L_s f) = I(f)$ for any $s \in G$, $f \in C_c^+(G)$. We will construct such a functional using the results on the functional I_φ from the previous section.

4.1. **Existence.** Functions $\Theta : C_c^+(G) \to \mathbb{R}_{>0}$ are naturally in one-to-one correspondence with the set

$$\mathbb{R}_{>0}^{C_c^+(G)} = \prod_{f \in C_c^+(G)} \mathbb{R}_{>0},$$

by corresponding Θ to the point $(\Theta(f))_{f \in C_c^+(G)}$. Fix some $f_0 \in C_c^+(G)$, and for any $\varphi \in C_c^+(G)$, we consider each I_φ as an element in the set above. Then by the first part of Proposition 3.1, we have that each I_φ satisfies

$$I_\varphi \in Y = \prod_{f \in C_c^+(G)} [(f_0 : f)^{-1}, (f : f_0)],$$

where Y is compact by Tychonoff's theorem. For any compact neighborhood K of the identity in G, consider the set $M_K = \{I_\varphi \mid \mathrm{supp}(\varphi) \subset K\} \subset Y$. Since Y is compact and thus closed, we also have $\overline{M}_K \subset Y$. For any finite number $K_1, K_2, \ldots, K_n$ of compact neighborhoods of 1 in G, we have $M_{\cap_{j=1}^n K_j}$ is nonempty by Theorem 1.2. We also have

$$M_{\cap_{j=1}^n K_j} \subset \cap_{j=1}^n M_{K_j} \subset \cap_{j=1}^n \overline{M}_{K_j},$$

and so any finite number of sets in the collection $\{\overline{M}_K\}$ of closed sets has nonempty intersection, or has the finite intersection property. Since each $\overline{M}_K \subset Y$ and Y is compact, then we have $\cap_K \overline{M}_K$ is nonempty. Let $\tilde{I} \in \cap_K \overline{M}_K \subset Y$, $\tilde{I} : C_c^+(G) \to \mathbb{R}_{>0}$ be an element in this nonempty intersection.

Since $\tilde{I} \in \cap_K \overline{M}_K$, then every neighborhood of $\tilde{I}$ in Y intersects each M_K. In particular, given any compact neighborhood K of 1 in G, any three functions $f_1, f_2, f_3 \in C_c^+(G)$ (or any finite number, from the product topology, but we will need at most three), and any $\varepsilon > 0$, there is a $\varphi \in C_c^+(G)$ such that $\mathrm{supp}(\varphi) \subset K$ and $|\tilde{I}(f_j) - I_\varphi(f_j)| < \varepsilon$ for $j = 1, 2, 3$.

Now let $f \in C_c^+(G)$, $c \in \mathbb{R}_{>0}$, and $\varepsilon > 0$. Let $\delta = \min\{\varepsilon/2, \varepsilon/2c\}$, and K some compact neighborhood of 1 in G. Then there is a $\varphi \in C_c^+(G)$ with $\mathrm{supp}(\varphi) \subset K$ such that

$$|\tilde{I}(cf) - I_\varphi(cf)| < \delta \le \varepsilon/2 \quad \text{and} \quad |\tilde{I}(f) - I_\varphi(f)| < \delta \le \varepsilon/2c.$$

So, $|c\tilde{I}(f) - I_\varphi(cf)| < \varepsilon/2$, since $cI_\varphi(f) = I_\varphi(cf)$, by part (1) of Lemma 3.1. Now we have, for any $f \in C_c^+(G)$, $c > 0$, and $\varepsilon > 0$, $|\tilde{I}(cf) - c\tilde{I}(f)| < \varepsilon$ by the triangle inequality, and so $\tilde{I}(cf) = c\tilde{I}(f)$.

Similarly, if $f \in C_c^+(G)$, $s \in G$, $\varepsilon > 0$, and K some compact neighborhood of 1 in G, there is a $\varphi \in C_c^+(G)$ with $\mathrm{supp}(\varphi) \subset K$ such that

$$|\tilde{I}(f) - I_\varphi(f)| < \varepsilon/2 \quad \text{and} \quad |\tilde{I}(L_s f) - I_\varphi(L_s f)| < \varepsilon/2.$$

By part (2) of Lemma 3.1, we have $I_\varphi(f) = I_\varphi(L_s f)$, and so the triangle inequality gives $|\tilde{I}(f) - \tilde{I}(L_s f)| < \varepsilon$ for any $f \in C_c^+(G)$, $s \in G$, and $\varepsilon > 0$. Thus $\tilde{I}(f) = \tilde{I}(L_s f)$.

Let $f_1, f_2 \in C_c^+(G)$, and let $\varepsilon > 0$. Applying Proposition 3.1, let V be a neighborhood of 1 in G such that, if $\varphi \in C_c^+(G)$ with $\mathrm{supp}(\varphi) \subset V$, then

$$I_\varphi(f_1) + I_\varphi(f_2) \leq I_\varphi(f_1 + f_2) + \varepsilon/4.$$

By (3.1), we also have $I_\varphi(f_1+f_2) \leq I_\varphi(f_1)+I_\varphi(f_2)$. If we let K be a compact neighborhood of 1 such that $K \subset V$, then for any $\varphi \in C_c^+(G)$ with $\mathrm{supp}(\varphi) \subset K$, we have

$$|I_\varphi(f_1 + f_2) - I_\varphi(f_1) - I_\varphi(f_2)| < \varepsilon/4.$$

Now let $\varphi \in C_c^+(G)$ with $\mathrm{supp}(\varphi) \subset K$ such that

$$|I_\varphi(f_i) - \tilde{I}(f_i)| < \varepsilon/4 \text{ for } i = 1, 2, \text{ and } |I_\varphi(f_1 + f_2) - \tilde{I}(f_1 + f_2)| < \varepsilon/4.$$

From the triangle inequality, we now have that for any $f_1, f_2 \in C_c^+(G)$ and $\varepsilon > 0$,

$$|\tilde{I}(f_1 + f_2) - (\tilde{I}(f_1) + \tilde{I}(f_2))| < \varepsilon,$$

and so $\tilde{I}(f_1 + f_2) = \tilde{I}(f_1) + \tilde{I}(f_2)$.

We have now shown that there is a function $\tilde{I} : C_c^+(G) \to \mathbb{R}_{>0}$ such that $\tilde{I}(cf) = c\tilde{I}(f)$, $\tilde{I}(L_s f) = \tilde{I}(f)$, and $\tilde{I}(f_1 + f_2) = \tilde{I}(f_1) + \tilde{I}(f_2)$, for any $f, f_1, f_2 \in C_c^+(G)$, $c \in \mathbb{R}_{>0}$, and $s \in G$. We extend $\tilde{I}$ to a left translation invariant linear functional I on all of $C_c(G)$ as follows. Define I to be 0 on the constant 0 function, and for any $f \in C_c(G)$, write $f = u+iv$ where u and v are real-valued, and define $I(f) = \tilde{I}(u^+)-\tilde{I}(u^-)+i\tilde{I}(v^+)-i\tilde{I}(v^-)$. The fact that I is a left translation invariant linear functional on $C_c(G)$ follows from the proven properties of $\tilde{I}$ (Exercise). The existence of such an I implies the existence of a left Haar measure on G.

4.2. **Uniqueness.** Let μ and ν be two left Haar measures on G. If $\psi \in C_c^+(G)$, then let $I(\psi) = \int_G \psi \, d\mu$ and $J(\psi) = \int_G \psi \, d\nu$. To show that ν is a scalar multiple of μ, we must show that $I(\psi)/J(\psi)$ is independent of $\psi \in C_c^+(G)$. Let $\psi, \theta \in C_c^+(G)$, and $\varepsilon > 0$. We show that $|I(\psi)/J(\psi) - I(\theta)/J(\theta)| < \varepsilon$ by finding a $\xi \in C_c^+(G)$ such that $I(\psi)/J(\psi)$ and $I(\theta)/J(\theta)$ are simultaneously made arbitrarily close to $I(\xi)/J(\xi)$.

Let K be a compact neighborhood of 1 in G, and let U be an open symmetric neighborhood of 1, $U \subset K$, and then the closure of U, $\overline{U} = K_0$, is compact, and K_0 is also symmetric since K is. Now let

$$K_\psi = \mathrm{supp}(\psi) \cdot K_0 \cup K_0 \cdot \mathrm{supp}(\psi) \quad \text{and} \quad K_\theta = \mathrm{supp}(\theta) \cdot K_0 \cup K_0 \cdot \mathrm{supp}(\theta).$$

Then K_ψ and K_θ are both compact. For any $t \in K_0$, define $\gamma_t\psi$ and $\gamma_t\theta$ by

$$\gamma_t\psi = R_t\psi - L_{t^{-1}}\psi \quad \text{and} \quad \gamma_t\theta = R_t\theta - L_{t^{-1}}\theta.$$

Notice that if $t \in K_0$ and $x \notin K_\psi$, then in particular $x \notin \mathrm{supp}(\psi)\, t^{-1}$ and $x \notin t^{-1} \mathrm{supp}(\psi)$, since K_0 is symmetric. That is, if $x \notin K_\psi$, then $\gamma_t\psi(x) = \psi(xt) - \psi(tx) = 0$, and so

$\text{supp}(\gamma_t\psi) \subset K_\psi$, and similarly $\text{supp}(\gamma_t\theta) \subset K_\theta$, for any $t \in K_0$. In particular, for any $t \in G$, $\gamma_t\psi$ and $\gamma_t\theta$ are both real-valued functions in $C_c(G)$.

We claim that for any $\delta > 0$, there is a compact symmetric neighborhood K_1 of the identity in G, with $K_1 \subset K_0$, such that $|\gamma_t\psi(s)| < \delta$ and $|\gamma_t\theta(s)| < \delta$ for all $t \in K_1$, $s \in G$. First, from right uniform continuity of ψ, find a neighborhood V_1 of the identity such that $t \in V_1$ implies $||R_t\psi - \psi||_\infty < \delta/2$. From left uniform coninuity, find a symmetric neighborhood V_2 of the identity such that $t \in V_2$ implies $||L_{t^{-1}}\psi - \psi||_\infty < \delta/2$. Now let V_0 be an open neighborhood of the identity contained in $V_1 \cap V_2 \cap K_0$, so that $V_0 \subset K_0$, and by the triangle inequality we have $t \in V$ implies $||\gamma_t\psi||_\infty < \delta$. Similarly, we can find an open neighborhood W_0 of the identity such that $W_0 \subset K_0$, and $t \in W_0$ implies $||\gamma_t\theta||_\infty < \delta$. Letting $U_0 = V_0 \cap W_0$, we have $U_0 \subset K_0$, and $t \in U_0$ guarantees that $||\gamma_t\psi||_\infty$ and $||\gamma_t\theta||_\infty$ are both less that δ. Now let K' be a compact neighborhood of the identity with $K' \subset U_0$, which exists by local compactness, and let U_1 be a symmetric open neighborhood of the identity contained in K'. Since $U_1 \subset K_0$ and K_0 is compact, then $K_1 = \overline{U}_1$ is a compact symmetric neighbhorhood of the identity, and $K_1 \subset K_0$. Moreover, since $K_1 \subset K' \subset U_0$, then $t \in K_1$ implies that $|\gamma_t\psi(s)| < \delta$ and $|\gamma_t\theta(s)| < \delta$ for all $s \in G$.

We let $\delta > 0$ be such that

$$\delta < \min\left\{\frac{\varepsilon J(\psi)}{2\mu(K_\psi)}, \frac{\varepsilon J(\theta)}{2\mu(K_\theta)}\right\}, \tag{4.1}$$

and we choose a compact symmetric neighborhood K_1 of the identity with $K_1 \subset K_0$ as above for this δ. Note that $\mu(K_\psi)$ and $\mu(K_\theta)$ are finite since the measure of a compact set is finite, and $J(\psi)$ and $J(\theta)$ are finite since ψ and θ have compact support.

Now let K_2 be a compact neighborhood of the identity such that $K_2 \subset \text{int}(K_1)$, which exists by local compactness. By Theorem 1.2, there is a function $\tilde{\xi} \in C_c^+(G)$ such that $\tilde{\xi}(g) = 1$ if $g \in K_2$, $\tilde{\xi}(g) = 0$ if $g \notin \text{int}(K_1)$, and $||\tilde{\xi}||_\infty = 1$. Define ξ by

$$\xi(s) = \tilde{\xi}(s) + \tilde{\xi}(s^{-1}),$$

so that $\text{supp}(\xi) \subset K_1$, since if $s \notin K_1$, then $s^{-1} \notin K_1$ because K_1 is symmetric, and by choice of $\tilde{\xi}$ we then have $\xi(s) = 0$. Thus $\xi \in C_c^+(G)$, and ξ is *even* in that $\xi(s) = \xi(s^{-1})$ for every $s \in G$.

We have

$$I(\psi)J(\xi) = \left(\int_G \psi(s)\, d\mu(s)\right)\left(\int_G \xi(t)\, d\nu(t)\right) = \int_G\int_G \psi(s)\xi(t)\, d\mu(s)d\nu(t),$$

and since μ is a left Haar measure, we have

$$I(\psi)J(\xi) = \int_G\int_G \psi(ts)\xi(t)\, d\mu(s)d\nu(t). \tag{4.2}$$

We also have, since μ is a left Haar measure,

$$I(\xi)J(\psi) = \int_G\int_G \xi(s)\psi(t)\, d\mu(s)d\nu(t) = \int_G\int_G \xi(t^{-1}s)\psi(t)\, d\mu(s)d\nu(t).$$

Since ξ is even and ν is a left Haar measure, we obtain that

$$I(\xi)J(\psi) = \int_G\int_G \xi(s^{-1}t)\psi(t)\, d\mu(s)d\nu(t) = \int_G\int_G \xi(t)\psi(st)\, d\mu(s)d\nu(t). \tag{4.3}$$

Calculating the absolute value of the difference between (4.2) and (4.3), we get

$$|I(\xi)J(\psi) - I(\psi)J(\xi)| = \left| \int_G \int_G \xi(t)\gamma_t\psi(s)\; d\mu(s)d\nu(t) \right|.$$

We have $\mathrm{supp}(\xi) \subset K_1$, $|\gamma_t\psi(s)| < \delta$ for every $s \in G$ and $t \in K_1$, and $\mathrm{supp}(\gamma_t\psi) \subset K_\psi$. So, we have

$$\begin{aligned} |I(\xi)J(\psi) - I(\psi)J(\xi)| &= \left| \int_{K_1} \xi(t) \left(\int_{K_\psi} \gamma_t\psi(s)\, d\mu(s) \right) d\nu(t) \right| \\ &\leq \int_{K_1} \xi(t) \left| \int_{K_\psi} \gamma_t\psi(s)\, d\mu(s) \right| d\nu(t) \\ &\leq \delta\mu(K_\psi) \int_{K_1} \xi(t)\; d\nu(t) = \delta\mu(K_\psi)J(\xi). \end{aligned}$$

Dividing both sides of this inequality by $J(\xi)J(\psi)$, and from our choice of δ (4.1), gives

$$\left| \frac{I(\xi)}{J(\xi)} - \frac{I(\psi)}{J(\psi)} \right| \leq \delta \frac{\mu(K_\psi)}{J(\psi)} < \varepsilon/2.$$

Duplicating this exact calculation, but replacing ψ by θ, yields

$$\left| \frac{I(\xi)}{J(\xi)} - \frac{I(\theta)}{J(\theta)} \right| \leq \delta \frac{\mu(K_\theta)}{J(\theta)} < \varepsilon/2,$$

and so $|I(\psi)/J(\psi) - I(\theta)/J(\theta)| < \varepsilon$ for any $\varepsilon > 0$. Thus, the linear functionals I and J differ by only a scalar factor, and the left Haar measure is unique up to scalar.

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