MATH 519 - PROBLEM SET 4

1. Let Γ be a totally disconnected compact group, and let $\pi : \Gamma \to \mathrm{GL}(n, \mathbb{C})$ be a homomorphism. Using the fact that the identity in $\mathrm{GL}(n, \mathbb{C})$ has a neighborhood which contains no non-trivial subgroup of $\mathrm{GL}(n, \mathbb{C})$, prove that π is continuous if and only if $\ker(\pi)$ is an open subgroup of Γ.

2. Let G be a unimodular locally compact totally disconnected group, and let V be a smooth $\mathcal{H}(G)$-module. Show that there is a smooth representation (π, V) of G such that $\pi(f)v = f \cdot v$ for every $f \in \mathcal{H}(G)$, as follows. If $g \in G$ and $v \in V$, let K_0 be a compact open subgroup of G such that $v \in \epsilon_{K_0} \cdot V$, which exists since V is a smooth $\mathcal{H}(G)$-module. Define $\pi(g)v = \epsilon_{gK_0} \cdot v$.
(a): Given $g \in G$, $v \in V$, and K_0 as above, let $K_1 \subset K_0$ be another compact open subgroup. First show that $v \in \epsilon_{K_1} \cdot V$. Then, show that $\epsilon_{gK_1} * \epsilon_{K_0} = \epsilon_{gK_0}$, and use this to conclude that the definition of $\pi(g)v$ does not depend on the choice of K_0.
(b): Let $g_1 \in G$, $v \in V$. Let K_1 be a compact open subgroup of G such that $v \in \epsilon_{K_1} \cdot V$, and define $K_2 = g_1 K_1 g_1^{-1}$. Show that $\pi(g_1)v \in \epsilon_{K_2} \cdot V$.
(c): Let $g_1, g_2 \in G$, $v \in V$, and let K_1 and K_2 be as in part **(b)**. Prove that $\epsilon_{g_2 K_2} * \epsilon_{g_1 K_1} = \epsilon_{g_2 g_1 K_1}$, and conclude that $\pi(g_2 g_1)v = \pi(g_2)\pi(g_1)v$, so that $\pi : G \to \mathrm{Aut}(V)$ is a homomorphism.
(d): Prove that the (π, V) which has been contructed is a smooth representation of G.
(e): Given any $f \in \mathcal{H}(G)$ and $v \in V$, choose a compact open subgroup K_0 of G such that $f \in \mathcal{H}_{K_0}(G)$ and $v \in \epsilon_{K_0} \cdot V$. Show that f is a linear combination of functions of the form $\epsilon_{g_i K_0}$, and use this to show $\pi(f)v = f \cdot v$.

3. Let G be a locally compact totally disconnected group, ι an order 2 continuous automorphism of G, and let (π, V) be an irreducible admissible representation of G. For $g \in G$ and $v \in V$, define ${}^\iota\pi$ by ${}^\iota\pi(g)v = \pi({}^\iota g)v$.
(a): Show that $({}^\iota\pi, V)$ is an irreducible admissible representation of G.
(b): Let $(\hat{\pi}, \hat{V})$ be the contragredient representation of (π, V). Prove that ${}^\iota\pi \cong \hat{\pi}$ if and only if there exists a nondegenerate bilinear form $B : V \times V \to \mathbb{C}$ such that

$$B(\pi(g)v, {}^\iota\pi(g)w) = B(v, w) \quad \text{for all } g \in G, v, w \in V.$$

(c): Assume that ${}^\iota\pi \cong \hat{\pi}$. Use Schur's Lemma to show that the bilinear form B from part **(b)** is unique up to scalar multiple.
(d): Continuing from part **(c)**, by considering the bilinear form B' defined by $B'(v, w) = B(w, v)$, prove that the bilinear form B must be either symmetric or skew-symmetric. That is, show that either $B(v, w) = B(w, v)$ for every $v, w \in V$, or $B(v, w) = -B(w, v)$ for every $v, w \in V$.