# MATH 519 - PROBLEM SET 3

**1.** Let $G$ be a locally compact group, and $\mu$ a left Haar measure on $G$. Show that $\mu$ is positive on all nonempty open subsets of $G$, and for every $f \in C_c^+(G)$, we have $\int_G f \, d\mu > 0$.

**2.** Let $F$ be a non-archimedean local field with ring of integers $\mathcal{O}$, uniformizer $\pi$, with $\mathfrak{p} = \pi\mathcal{O}$, and suppose the absolute value $|\cdot|_v$ on $F$ is normalized so that $|\pi|_v = q^{-1}$, where $q$ is the cardinality of the residue field $\mathcal{O}/\mathfrak{p}$. Let $\mu$ be the additive Haar measure on $F$, normalized so that $\mu(\mathcal{O}) = 1$. Prove that the multiplicative Haar measure on $F^\times$ is given by $\int \ |x|_v^{-1} \, d\mu(x)$.

**3.** Let $F$ be a non-archimedean local field, with the same notations and normalizations as in Problem 2. Let $\int \ d\mu(X)$ denote the additive Haar measure on the set $M_n(F)$ of $n$-by-$n$ matrices over $F$, so that $\int \ d\mu(X) = \int \ d\mu(X_{11})d\mu(X_{12}) \cdots d\mu(X_{nn})$, where $X = (X_{ij})$, and $\int \ d\mu(X_{ij})$ is the additive Haar measure on $F$. Prove that $\int \ |\det(X)|_v^{-n} \, d\mu(X)$ is a left and right Haar measure on the group $\mathrm{GL}(n, F)$. Hint: Check the invariance of the measure by left and right multiplication of each type of elementary matrix. To check this invariance, look at each coordinate and use the fact that we have the additive Haar measure on $F$ and the multiplicative Haar measure on $F^\times$. Also, use Fubini's Theorem to change the order of integration when necessary.

**4.** Let $F$ be a non-archimedean local field, with the same conventions as above. Let $N(n, F)$ denote the group of unipotent $n$-by-$n$ matrices over $F$. Using the same method as in Problem 3, show that $\int \prod_{i<j} d\mu(X_{ij}) = \int \ d\mu(X_{12})d\mu(X_{13}) \cdots d\mu(X_{n-1,n})$ is a left and right Haar measure for the group $N(n, F)$.

**5.** Let $F$ be a non-archimedean local field, continuing all above notations. Let $G = B(2, F)$ be the standard Borel subgroup of $\mathrm{GL}(2, F)$, that is, the group of upper triangular 2-by-2 matrices. Writing elements of $B(2, F)$ in the form $X = \begin{pmatrix} x_1 & y \\ 0 & x_2 \end{pmatrix}$, show that the left Haar measure of $B(2, F)$ is given by $\int \ |x_1|_v^{-2} \, |x_2|_v^{-1} \, d\mu(x_1)d\mu(x_2)d\mu(y)$, the right Haar measure is given by $\int \ |x_1|_v^{-1} \, |x_2|_v^{-2} \, d\mu(x_1)d\mu(x_2)d\mu(y)$, and the modular quasicharacter is $\delta_G(X) = |x_1|_v |x_2|_v^{-1}$. Hint: Check invariance under multiplication by elements of the form $\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$ and $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$.