MATH 519 - PROBLEM SET 1

1. (a): Let H be a subgroup of the topological group G. Show that if H is a normal subgroup of G, then $\overline{H}$ is also a normal subgroup of G.
(b): Suppose G is a T_1 topological group. Show that if H is an abelian subgroup of G, then $\overline{H}$ is also an abelian subgroup of G. Find a counterexample to the statement if we remove the condition that G is T_1. (Hint: Start with a nonabelian group and give it a weak topology.)

2. Let G be a topological group, let 1_G denote the identity element in G, and let F be any closed subset in G such that $1_G \notin F$.
(a): Define $U_0 = G \setminus F$, and define U_i, $i \geq 1$ recursively by letting U_i be a symmetric neighborhood of 1_G such that $U_i^2 \subset U_{i-1} \cap U_0$. Now, for every dyadic rational number $r = k/2^n$, $n \geq 0$, $1 \leq k \leq 2^n$, we define a set V_r as follows:

$$V_{1/2^n} = U_n, \quad V_{2k/2^{n+1}} = V_{k/2^n}, \quad \text{and} \quad V_{(2k+1)/2^{n+1}} = V_{1/2^{n+1}} V_{k/2^n}.$$

Prove that the definition of V_r does not depend on the representation of r as a dyadic fraction, and $V_r \subset V_{r'}$ whenever $r < r' \leq 1$.
(b): Define $f : G \to [0,1]$ as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \in V_r \text{ for all } r, \\ 1 & \text{if } x \notin V_1, \\ \sup\{r \mid r \leq 1, x \notin V_r\} & \text{otherwise.} \end{cases}$$

Then $f(1_G) = 0$ and $f(h) = 1$ for every $h \in F$, since $V_1 = U_0 = G \setminus F$. Prove that f is continuous. (Hint: Let $\varepsilon > 0$, and let n be such that $1/2^n < \varepsilon$. First suppose $f(x) < 1$, and choose integers m and k such that $k > n$, $m < 2^k$, and such that $x \in V_{m/2^k} \setminus V_{(m-1)/2^k}$. Consider $y \in V_{1/2^k} x$, and show that we must then have $x \in V_{1/2^k} y$. Now bound the possible value of $f(y)$. Treat the case $f(x) = 1$ similarly.)

3. (a): Let G be a topological group with subgroup H. Show that H is open if and only if G/H is a discrete space.
(b): Let G be a compact group with subgroup H. Show that H is open if and only if G/H is finite and discrete.

4. Let G and H be topological groups, and let $\varphi : G \to H$ be a surjective continuous homomorphism. Show that there is a unique injective continuous homomorphism $\bar{\varphi} : G/\ker(\varphi) \to H$ such that $\varphi = \bar{\varphi} \circ p$, where $p : G \to G/\ker(\varphi)$ is the natural projection map.

5. (a): Let G be a topological group and G° the connected component of the identity of G. Show that G° is a normal subgroup of G, and that the connected components of G are of the form xG°.
(b): Show that G/G° is a totally disconnected group. (Hint: Let S be a connected component of G/G°, and show that $p^{-1}(S)$ must be a coset of G°, where $p : G \to G/G^\circ$ is the natural projection map.)