Math 103 - HW #4 A, B

A1. $4^{1/4} \cdot \left(\frac{1}{2}\right)^3 = \left(2^{1/4}\right) \cdot \left(2^{-1}\right)^3 = 2^{2/4} \cdot 2^3 = 2^{1/2} \cdot 2^3 = 2^{7/2}$

A2. $\frac{9^{1/6} \cdot 3^{1/2}}{3^{-3} \cdot 3^4} = \frac{(3^2)^{1/6} \cdot 3^{1/2}}{3^{-3+4}} = \frac{3^{2/6} \cdot 3^{1/2}}{3^1} = 3^{1/3} \cdot 3^{1/2} \cdot 3^{-1} = 3^{2/6} \cdot \frac{3}{6-1} = 3^{3/6-1} = \sqrt[6]{3^{-16}}$

A3. $(1000)^{1/3} \cdot 10^7 \cdot (10^2)^{1/2} \cdot (100)^{-11/2}$

$= (10^3)^{1/3} \cdot 10^7 \cdot 10^6 \cdot (10^2)^{-1/2} = 10^1 \cdot 10^7 \cdot 10^6 \cdot 10^{-1}$

$= 10^{13}$

A4. $\sqrt[5]{b^4} \cdot (b^2)^{3/5} \cdot b^{-3} = (b^{4/5}) \cdot (b^{2 \cdot 3/5}) \cdot b^{-3}$

$= b^{4/5} \cdot b^{6/5} \cdot b^{-3} = b^{10/5} \cdot b^{-3} = b^2 \cdot b^{-3} = b^{-1}$

If $b = 0$, then $b^{-1} = \frac{1}{b}$ is undefined.
\[
A5. \left( \frac{3\sqrt[6]{(x+1)^2}}{\sqrt[3]{(x+1)^3}} \right)^2 = \left( \left( \frac{(x+1)^2}{(x+1)^3} \right)^{1/3} \right)^2 = \left( (x+1)^{-1/3} \right)^2 = (x+1)^{-2/3} = \left( (x+1)^{3/2} - 3/2 + 1/3 \right)^2 = \left( (x+1)^{-1/2} \right)^2 = \frac{1}{(x+1)}
\]

A6. We have \((x^2 - x - 6)^{1/4} = \sqrt[4]{x^2 - x - 6}\). The 4th root (or any even root) is only defined for non-negative numbers (otherwise the expression is not a real number). That is, \(\sqrt[4]{x^2 - x - 6}\) is defined exactly when \(x^2 - x - 6 > 0\), so when \((x-3)(x+2) > 0\). This is true when either \(x-3 > 0\) and \(x+2 > 0\), or, \(x-3 \leq 0\) and \(x+2 \leq 0\), so when either \(x > 3\) and \(x > -2\), or, \(x \leq 3\) and \(x \leq -2\). This means when \(x > 3\) or \(x < -2\). So the expression \((x^2 - x - 6)^{1/4}\) is defined when \(x > 3\) or \(x < -2\). On the number line:

\[\boxed{-2 \quad 0 \quad 3}\]
A7. Note that \((x^{1/2})^2 = x\), so we can write the equation \(x - 2x^{1/2} - 8 = 0\) as \((x^{1/2})^2 - 2x^{1/2} - 8 = 0\). If we think of \(z = x^{1/2}\), this becomes \(z^2 - 2z - 8 = 0\), or \((z - 4)(z + 2) = 0\). That is, the original equation factors as \((x^{1/2} - 4)(x^{1/2} + 2) = 0\), or \((\sqrt{x} - 4)(\sqrt{x} + 2) = 0\). This expression is 0 when \(\sqrt{x} - 4 = 0\) or \(\sqrt{x} + 2 = 0\), so when \(\sqrt{x} = 4\) or \(\sqrt{x} = -2\). \(\sqrt{x} = 4\) when \((\sqrt{x})^2 = 4^2\), so \(x = 16\). However, \(\sqrt{x} = -2\) cannot occur, because \(\sqrt{x}\) means the positive square root of \(x\), so \(\sqrt{x} \geq 0\) always (that is, \(\sqrt{x}\) is never negative). So, the only solution is \(\boxed{x = 16}\).

A8. (a) Similar to A6, the expression \(\sqrt{x-2}\) is only defined when \(x-2 \geq 0\), since we are taking an even root. The expression is only defined (as a real number) for \(x \geq 2\), so for example if \(x = 1\), then \(\sqrt{x-2} = \sqrt{1-2}\) is not a real number. So the statement is \(\text{FALSE}\).
A8 (b): We compute the expression under the square root:
\[
(x^{1/2} - x^{-1/2})^2 + 4 = (x^{1/2} - x^{-1/2})(x^{1/2} - x^{-1/2}) + 4
\]
\[
= (x^{1/2})^2 - (x^{-1/2})(x^{1/2}) - (x^{1/2})(x^{-1/2}) + (-x^{-1/2})^2 + 4
\]
\[
= x - x^{-1/2} - x^{1/2} + x^{-1} + 4 = x - x^0 - x^0 + x^{-1} + 4
\]
\[
= x - 2 + x^{-1} + 4 = x + 2 + x^{-1}.
\]

If we square the right side, we have:
\[
(x^{1/2} + x^{-1/2})^2 = (x^{1/2} + x^{-1/2})(x^{1/2} + x^{-1/2}) =
\]
\[
= (x^{1/2})^2 + (x^{-1/2})(x^{1/2}) + (x^{1/2})(x^{-1/2}) + (x^{-1/2})^2
\]
\[
= x^1 + x^0 + x^0 + x^{-1} = x + 2 + x^{-1}.
\]

But now we have \((x^{1/2} + x^{-1/2})^2 = (x^{1/2} - x^{-1/2})^2 + 4\),
since they are both equal to \(x + 2 + x^{-1/2}\).

Taking the (positive) square root of both sides gives
\[
\sqrt{(x^{1/2} - x^{-1/2})^2 + 4} = x^{1/2} + x^{-1/2},
\]
so the statement

is \textbf{TRUE}.
B1. (a) \(100^{3/2} = 1000\) means \(\log_{100} 1000 = \frac{3}{2}\).

(b) \((\frac{1}{2})^{-3} = 8\) means \(\log_{\frac{1}{2}} 8 = -3\).

(c) \((\frac{9}{16})^{-1/2} = \frac{4}{3}\) means \(\log_{\frac{9}{16}} \left(\frac{4}{3}\right) = -\frac{1}{2}\).

(d) \(\log_5 \sqrt[3]{25} = \frac{2}{3}\) means \(5^{2/3} = \sqrt[3]{25}\).

(e) \(\log_{100} 0.001 = -\frac{3}{2}\) means \(100^{-3/2} = 0.001\).

(f) \(\log_{2/3} \left(\frac{27}{8}\right) = -3\) means \(\left(\frac{2}{3}\right)^{-3} = \frac{27}{8}\).

B2. (a) \(\log_2 \left(\frac{1}{16}\right) = \boxed{-4}\) since \(2^{-4} = \frac{1}{16} = \left(\frac{1}{2}\right)^4\).

(b) \(\log_3 \sqrt[5]{27} = \log_3 \sqrt[5]{3^3} = \log_3 (3^{3/5}) = \log_3 3^{3/5} = \boxed{\frac{3}{5}}\).

(c) \(\log_{10} 0.00001 = \log_{10} (10^{-5}) = \boxed{-5}\).

(d) \(\log_{3/4} \left(\sqrt[3]{\frac{16}{9}}\right) = \log_{3/4} \left(\sqrt[3]{\frac{4^2}{3^2}}\right) = \log_{3/4} \left(\frac{3}{4}\right)^{-2/3} = \boxed{-\frac{2}{3}}\).

(e) \(\log_2 (\frac{1}{\sqrt{2}}) = \log_2 (2^{-1/2}) = \log_2 (2^{-1/2}) = \boxed{-\frac{1}{2}}\).

(f) \(2 \log_x \sqrt{x} - \frac{1}{3} \log (y^3) = 2 \log_x (x^{1/2}) = \frac{1}{3} \cdot 3 = \frac{2}{3} - 1 = \frac{1}{3}\).

(g) \(\log_5 50 - 2 \log_5 2 + \log_5 10 = \log_5 \left(\frac{50 \cdot 10}{2^2}\right) = \log_5 \left(\frac{500}{2}\right) = \boxed{3}\).
B3. (a) \[ 4 \log_2 \left( x^{\frac{1}{2}} \right) - \frac{3}{2} \log_2 \left( x^{\frac{3}{4}} \right) + \log_2 x \]
\[ = \log_2 \left( \left( x^{\frac{1}{2}} \right)^4 \right) + \log_2 \left( \left( x^{\frac{3}{4}} \right)^{\frac{3}{2}} \right) + \log_2 x \]
\[ = \log_2 \left( x^{2} \right) + \log_2 \left( x^{\frac{9}{4}} \right) + \log_2 x \]
\[ = \log_2 \left( x^{2} \cdot x^{\frac{9}{4}} \cdot x \right) = \boxed{\log_2 \left( x^{\frac{13}{4}} \right)} \]

(b) \[ \log_3 \left( \frac{x^2}{y^3} \right) - 3 \log_3 \left( y^{-1} \right) - 2 \log_3 \left( \frac{y^{1/2}}{x^{-2}} \right) \]
\[ = \log_3 \left( \frac{x^2}{y^3} \right) + \log_3 \left( \left( y^{-1} \right)^{-3} \right) + \log_3 \left( \left( \frac{y^{1/2}}{x^{-2}} \right)^{-2} \right) \]
\[ = \log_3 \left( \frac{x^2}{y^3} \right) + \log_3 \left( y^3 \right) + \log_3 \left( \frac{y^{-1}}{x^4} \right) \]
\[ = \log_3 \left( \frac{x^2 \cdot y^3 \cdot y^{-1}}{x^4} \right) = \boxed{\log_3 \left( \frac{1}{xy} \right)} \]

B4. \( \log_3 \left( 2x - 4 \right) = 2 \) means \( 3^2 = 2x - 4 \), so
\( 2x - 4 = 9 \), so \( 2x = 13 \), so \( x = \frac{13}{2} \).

B5. \( 16 \cdot 2^x = 4^{13} \) can be written as:
\( 2^4 \cdot 2^x = (2^2)^{13} \), or \( 2^{4+x} = 2^{26} \). Taking \( \log_2 \) of both sides gives \( \log_2 (2^{4+x}) = \log_2 (2^{26}) \), so
\( 4+x=26 \), giving \( x=22 \).
B6. \( \log_2 (x^2 - 2x + 3) = 2 \) means \( 2^2 = x^2 - 2x + 3 \),
so \( x^2 - 2x + 3 = 4 \), or \( x^2 - 2x - 1 = 0 \).

We can't factor, so we use the quadratic formula
with \( a=1 \), \( b=-2 \), \( c=-1 \) (so \( b^2 - 4ac = 4 - 4(1)(-1) = 8 > 0 \)).

We get \( x = \frac{2 \pm \sqrt{4 - 4(1)(-1)}}{2} = \frac{2 \pm \sqrt{8}}{2} = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2} \).

The solutions are \( x = 1 + \sqrt{2} \) or \( 1 - \sqrt{2} \).

B7. \( b^{-7x} = 5 \cdot M \cdot y^3 \). To solve for \( x \), first take \( \log_b \) of both sides:

\[ \log_b (b^{-7x}) = \log_b (5 \cdot M \cdot y^3) \]

Since \( \log_b (b^{-7x}) = -7x \), we have \(-7x = \log_b (5 \cdot M \cdot y^3)\).

Thus \( x = -\frac{1}{7} \log_b (5 \cdot M \cdot y^3) \).

B8. If \( 2^y = 9 \), then \( (3^2)^y = 3^2 \), so \( 3^{2y} = 3^2 \). Taking \( \log_3 \) of both sides gives \( 3y = 2 \), so \( y = \frac{2}{3} \).

Now \( 8^{3x - 2y} = 4 \) means \( 2^{3(3x - 2y)} = 2^2 \), so

\( 9x - 6y = 2 \). Taking \( \log_2 \) of both sides gives

\( 9x - 6y = 2 \). Since \( y = \frac{2}{3} \), \( 9x - 6(\frac{2}{3}) = 2 \), so

\( 9x - 4 = 2 \), so \( 9x = 6 \), so \( x = \frac{2}{3} \).