Theorem The natural logarithm of a log normal(\(\alpha, \beta\)) random variable is a \(N(\mu, \sigma^2)\) random variable.

Proof Let the random variable \(X\) have the log normal distribution with probability density function
\[
f_X(x) = \frac{1}{\sqrt{2\pi x\beta}} e^{-\frac{1}{2}\left(\frac{\ln(x/\alpha)}{\beta}\right)^2} \quad x > 0.
\]
The transformation \(Y = g(X) = \ln(X)\) is a 1–1 transformation from \(\mathcal{X} = \{x \mid x > 0\}\) to \(\mathcal{Y} = \{y \mid -\infty < y < \infty\}\) with inverse \(X = g^{-1}(Y) = e^Y\) and Jacobian
\[
\frac{dX}{dY} = e^Y.
\]
Therefore, by the transformation technique, the probability density function of \(Y\) is
\[
f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = \frac{1}{\sqrt{2\pi e^y\beta}} e^{-\frac{1}{2}\left(\frac{\ln(e^y/\alpha)}{\beta}\right)^2} |e^y| = \frac{1}{\sqrt{2\pi \beta}} e^{-\frac{1}{2}\left(\frac{y-\ln(\alpha)}{\beta}\right)^2} \quad -\infty < y < \infty.
\]
Let \(\alpha = e^\mu\) and \(\beta = \sigma\). Then
\[
f_Y(y) = \frac{1}{\sqrt{2\pi \sigma}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} \quad -\infty < y < \infty,
\]
which is the probability density function of the normal distribution.

APPL verification: The APPL statements
\[
X := \text{LogNormalRV}(\mu, \sigma);
\]
\[
g := \{[x -> \ln(x)], [0, \infty]\};
\]
\[
Y := \text{Transform}(X, g);
\]
\[
Z := \text{NormalRV}(\mu, \sigma);
\]
yield identical functional forms
\[
f_Y(y) = \frac{1}{\sqrt{2\pi \sigma}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} \quad -\infty < y < \infty
\]
for the random variables \(Y\) and \(Z\), which verifies that the natural logarithm of a log normal random variable has the normal distribution.