example 12
Physical forces have magnitude and direction and may thus be represented by vectors. If several forces act at once on an object, the resultant force is represented by the sum of the individual force vectors. Suppose that forces \( \mathbf{i} + \mathbf{k} \) and \( \mathbf{j} + \mathbf{k} \) are acting on a body. What third force \( \mathbf{F} \) must we impose to counteract the two — that is, to make the total force equal to zero?

solution
The force \( \mathbf{F} \) should be chosen so that \( (\mathbf{i} + \mathbf{k}) + (\mathbf{j} + \mathbf{k}) + \mathbf{F} = \mathbf{0} \) that is, \( \mathbf{F} = -(\mathbf{i} + \mathbf{k}) - (\mathbf{j} + \mathbf{k}) = -\mathbf{i} - \mathbf{j} - 2\mathbf{k} \) (Recall that \( \mathbf{0} \) is the zero vector, the vector whose components are all zero.)

exercises

1. Calculate \((3\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} - \mathbf{k})\).

2. Calculate \(\mathbf{a} \cdot \mathbf{b}\), where \(\mathbf{a} = 2\mathbf{i} + 10\mathbf{j} - 12\mathbf{k}\) and \(\mathbf{b} = -3\mathbf{i} + 4\mathbf{k}\).

3. Find the angle between \(7\mathbf{j} + 19\mathbf{k}\) and \(-2\mathbf{i} - \mathbf{j}\) (to the nearest degree).

In Exercises 6 to 11, compute \(\|\mathbf{u}\|, \|\mathbf{v}\|, \) and \(\mathbf{u} \cdot \mathbf{v}\) for the given vectors in \(\mathbb{R}^3\).

4. Compute \(\mathbf{u} \cdot \mathbf{v}\) where \(\mathbf{u} = \sqrt{3}\mathbf{i} - 315\mathbf{j} + 22\mathbf{k}\) and \(\mathbf{v} = \mathbf{u}/\|\mathbf{u}\|\).

5. Is \(|8\mathbf{i} - 12\mathbf{k}| \cdot |6\mathbf{j} + \mathbf{k}| = |(8\mathbf{i} - 12\mathbf{k}) \cdot (6\mathbf{j} + \mathbf{k})|\) equal to zero? Explain.

6. \(\mathbf{u} = 15\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}\) \(\mathbf{v} = \pi \mathbf{i} + 3\mathbf{j} - \mathbf{k}\)

7. \(\mathbf{u} = 2\mathbf{j} - \mathbf{i}, \mathbf{v} = -\mathbf{j} + \mathbf{i}\)

8. \(\mathbf{u} = 5\mathbf{i} - \mathbf{j} + 2\mathbf{k}, \mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k}\)

9. \(\mathbf{u} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}, \mathbf{v} = -2\mathbf{i} - 3\mathbf{j} - 7\mathbf{k}\)

10. \(\mathbf{u} = -\mathbf{i} + 3\mathbf{k}, \mathbf{v} = 4\mathbf{j}\)

11. \(\mathbf{u} = -\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}, \mathbf{v} = -\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}\)

12. Let \(\mathbf{v} = (2, 3)\). Suppose \(\mathbf{w} \in \mathbb{R}^2\) is perpendicular to \(\mathbf{v}\), and that \(\|\mathbf{w}\| = 5\). This determines \(\mathbf{w}\) up to sign. Find one such \(\mathbf{w}\).

13. Find \(b\) and \(c\) so that \((5, b, c)\) is orthogonal to both \((1, 2, 3)\) and \((1, -2, 1)\).

14. Let \(\mathbf{v}_1 = (0, 3, 0), \mathbf{v}_2 = (2, 2, 0), \mathbf{v}_3 = (1, 1, 3)\). These three vectors with their tails at the origin determine a parallelepiped \(P\).
   (a) Draw \(P\).
   (b) Determine the length of the main diagonal (from the origin to its opposite vertex).

15. What is the geometric relation between the vectors \(\mathbf{v}\) and \(\mathbf{w}\) if \(\mathbf{v} \cdot \mathbf{w} = -\|\mathbf{v}\| \cdot \|\mathbf{w}\|\)?

16. Normalize the vectors in Exercises 6 to 8. (Only the solution corresponding to Exercise 7 is in the Student Guide.)

17. Find the angle between the vectors in Exercises 9 to 11. If necessary, express your answer in terms of \(\cos^{-1}\).

18. Find all values of \(x\) such that \((x, 1, x)\) and \((x, -6, 1)\) are orthogonal.

19. Find all values of \(x\) such that \((7, x, -10)\) and \((3, x, x)\) are orthogonal.

20. Find the projection of \(\mathbf{u} = -\mathbf{i} + \mathbf{j} + \mathbf{k}\) onto \(\mathbf{v} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}\).

21. Find the projection of \(\mathbf{v} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}\) onto \(\mathbf{u} = -\mathbf{i} + \mathbf{j} + \mathbf{k}\).

22. What restrictions must be made on the scalar \(b\) so that the vector \(2\mathbf{i} + b\mathbf{j}\) is orthogonal to \((a) -3\mathbf{i} + 2\mathbf{j} + \mathbf{k}\) and \((b) \mathbf{k}\)?

23. Vectors \(\mathbf{v}\) and \(\mathbf{w}\) are sides of an equilateral triangle whose sides have length 1. Compute \(\mathbf{v} \cdot \mathbf{w}\).

24. Let \(\mathbf{b} = (3, 1, 1)\) and \(P\) be the plane through the origin given by \(x + y + 2z = 0\).
   (a) Find an orthogonal basis for \(P\). That is, find two nonzero orthogonal vectors \(\mathbf{v}_1, \mathbf{v}_2 \in P\).
(b) Find the orthogonal projection of \( \mathbf{b} \) onto \( P \). That is, find \( \text{Proj}_{\mathbf{v}} \mathbf{b} + \text{Proj}_{\mathbf{w}} \mathbf{b} \).

25. Find two nonparallel vectors both orthogonal to \( (1, 1, 1) \).

26. Find the line through \( (3, 1, -2) \) that intersects and is perpendicular to the line 
   \[ x = -1 + t, \quad y = -2 + t, \quad z = -1 + t. \]  
   [Hint: If \( (x_0, y_0, z_0) \) is the point of intersection, find its coordinates.]

27. Using the dot product, prove the converse of the Pythagorean theorem. That is, show that if the lengths of the sides of a triangle satisfy \( a^2 + b^2 = c^2 \), then the triangle is a right triangle.

28. For \( \mathbf{v} = (v_1, v_2, v_3) \) let \( \alpha, \beta, \gamma \) denote the angles between \( \mathbf{v} \) and the \( x, y, \) and \( z \) axes, respectively. Show that \( \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 \).

29. A ship at position \( (1, 0) \) on a nautical chart (with north in the positive \( y \) direction) sights a rock at position \( (2, 4) \). What is the vector joining the ship to the rock? What angle \( \theta \) does this vector make with due north? (This is called the \textbf{bearing} of the rock from the ship.)

30. Suppose that the ship in Exercise 29 is pointing due north and traveling at a speed of 4 knots relative to the water. There is a current flowing due east at 1 knot. The units on the chart are nautical miles; 1 knot = 1 nautical mile per hour.
   (a) If there were no current, what vector \( \mathbf{u} \) would represent the velocity of the ship relative to the sea bottom?
   (b) If the ship were just drifting with the current, what vector \( \mathbf{v} \) would represent its velocity relative to the sea bottom?
   (c) What vector \( \mathbf{w} \) represents the total velocity of the ship?
   (d) Where would the ship be after 1 hour?
   (e) Should the captain change course?
   (f) What if the rock were an iceberg?

31. An airplane is located at position \( (3, 4, 5) \) at noon and traveling with velocity \( 400 \mathbf{i} + 500 \mathbf{j} - \mathbf{k} \) kilometers per hour. The pilot spots an airport at position \( (23, 29, 0) \).
   (a) At what time will the plane pass directly over the airport? (Assume that the plane is flying over flat ground and that the vector \( \mathbf{k} \) points straight up.)
   (b) How high above the airport will the plane be when it passes?

32. The wind velocity \( \mathbf{v}_1 \) is 40 miles per hour (mi/h) from east to west while an airplane travels with air speed \( \mathbf{v}_2 \) of 100 mi/h due north. The speed of the airplane relative to the ground is the vector sum \( \mathbf{v}_1 + \mathbf{v}_2 \).
   (a) Find \( \mathbf{v}_1 + \mathbf{v}_2 \).
   (b) Draw a figure to scale.

33. A force of 50 lb is directed 50\(^\circ\) above horizontal, pointing to the right. Determine its horizontal and vertical components. Display all results in a figure.

34. Two persons pull horizontally on ropes attached to a post, the angle between the ropes being 60\(^\circ\). Person A pulls with a force of 150 lb, while person B pulls with a force of 110 lb.
   (a) The resultant force is the vector sum of the two forces. Draw a figure to scale that graphically represents the three forces.
   (b) Using trigonometry, determine formulas for the vector components of the two forces in a conveniently chosen coordinate system. Perform the algebraic addition, and find the angle the resultant force makes with \( \mathbf{A} \).

35. A 1-kilogram (1-kg) mass located at the origin is suspended by ropes attached to the two points \( (1, 1, 1) \) and \( (-1, -1, 1) \). If the force of gravity is pointing in the direction of the vector \(-\mathbf{k}\), what is the vector describing the force along each rope? [Hint: Use the symmetry of the problem. A 1-kg mass weighs 9.8 newtons (N).]

36. Suppose that an object moving in direction \( \mathbf{i} + \mathbf{j} \) is acted on by a force given by the vector \( 2\mathbf{i} + \mathbf{j} \). Express this force as the sum of a force in the direction of motion and a force perpendicular to the direction of motion.

37. A force of 6 N makes an angle of \( \pi/4 \) radian with the \( y \) axis, pointing to the right. The force acts against the movement of an object along the straight line connecting \( (1, 2) \) to \( (5, 4) \).
   (a) Find a formula for the force vector \( \mathbf{F} \).
   (b) Find the angle \( \theta \) between the displacement direction \( \mathbf{D} = (5 - 1)\mathbf{i} + (4 - 2)\mathbf{j} \) and the force direction \( \mathbf{F} \).
   (c) The \textbf{work done} is \( \mathbf{F} \cdot \mathbf{D} \), or, equivalently, \( \|\mathbf{F}\| \|\mathbf{D}\| \cos \theta \). Compute the work from both formulas and compare.
38. Show that in any parallelogram the sum of the squares of the lengths of the four sides equals the sum of the squares of the lengths of the two diagonals.

39. Using vectors, show that the diagonals of a rectangle are perpendicular if and only if the rectangle is a square.

1.3 Matrices, Determinants, and the Cross Product

In Section 1.2 we defined a product of vectors that was a scalar. In this section we shall define a product of vectors that is a vector; that is, we shall show how, given two vectors \( \mathbf{a} \) and \( \mathbf{b} \), we can produce a third vector \( \mathbf{a} \times \mathbf{b} \), called the cross product of \( \mathbf{a} \) and \( \mathbf{b} \). This new vector will have the pleasing geometric property that it is perpendicular to the plane spanned (determined) by \( \mathbf{a} \) and \( \mathbf{b} \). The definition of the cross product is based on the notions of the matrix and the determinant, and so these are developed first. Once this has been accomplished, we can study the geometric implications of the mathematical structure we have built.

2 \( \times \) 2 Matrices

We define a 2 \( \times \) 2 matrix to be an array

\[
\begin{bmatrix}
    a_{11} & a_{12} \\
    a_{21} & a_{22}
\end{bmatrix},
\]

where \( a_{11}, a_{12}, a_{21}, \) and \( a_{22} \) are four scalars. For example,

\[
\begin{bmatrix}
    2 & 1 \\
    0 & 4
\end{bmatrix}, \quad
\begin{bmatrix}
    -1 & 0 \\
    1 & 1
\end{bmatrix}, \quad \text{and} \quad
\begin{bmatrix}
    13 & 7 \\
    6 & 11
\end{bmatrix}
\]

are 2 \( \times \) 2 matrices. The determinant

\[
\begin{vmatrix}
    a_{11} & a_{12} \\
    a_{21} & a_{22}
\end{vmatrix}
\]

of such a matrix is the real number defined by the equation

\[
\begin{vmatrix}
    a_{11} & a_{12} \\
    a_{21} & a_{22}
\end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.
\]

\[\text{(1)}\]

example \ 1

\[
\begin{bmatrix}
    1 & 1 \\
    1 & 1
\end{bmatrix} = 1 - 1 = 0; \quad
\begin{bmatrix}
    1 & 2 \\
    3 & 4
\end{bmatrix} = 4 - 6 = -2; \quad
\begin{bmatrix}
    5 & 6 \\
    7 & 8
\end{bmatrix} = 40 - 42 = -2
\]

3 \( \times \) 3 Matrices

A 3 \( \times \) 3 matrix is an array

\[
\begin{bmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{bmatrix},
\]
In 1879, Gibbs taught a course at Yale in vector analysis with applications to electricity and magnetism. This treatise was clearly motivated by the advent of Maxwell's equations, which we will be studying in Chapter 8. In 1884, he published his Elements of Vector Analysis, a book in which all the properties of the dot and cross products are fully developed. Knowing that much of what Gibbs wrote was in fact due to Tait, Gibbs's contemporaries did not view his book as highly original. However, it is one of the sources from which modern vector analysis has come into existence.

Heaviside was also largely motivated by Maxwell's brilliant work. His great Electromagnetic Theory was published in three volumes. Volume I (1893) contained the first extensive treatment of modern vector analysis.

We all owe a great debt to E. B. Wilson's 1901 book Vector Analysis: A Textbook for the Use of Students of Mathematics and Physics Founded upon the Lectures of J. Willard Gibbs. Wilson was reluctant to take Gibbs's course, because he had just completed a full-year course in quaternions at Harvard under J. M. Pierce, a champion of quaternionic methods; but he was forced by a dean to add the course to his program, and he did so in 1899. Wilson was later asked by the editor of the Yale Bicentennial Series to write a book based on Gibbs's lectures. For a picture of Gibbs and for additional historical comments on divergence and curl, see the Historical Note in Section 4.4.

exercises

1. Verify that interchanging the first two rows of the $3 \times 3$ determinant

$$
\begin{vmatrix}
1 & 2 & 1 \\
3 & 0 & 1 \\
2 & 0 & 2 \\
\end{vmatrix}
$$

changes the sign of the determinant.

2. Evaluate the determinants

(a) $\begin{vmatrix}
2 & -1 & 0 \\
4 & 3 & 2 \\
3 & 0 & 1 \\
\end{vmatrix}$

(b) $\begin{vmatrix}
36 & 18 & 17 \\
45 & 24 & 20 \\
3 & 5 & -2 \\
\end{vmatrix}$

(c) $\begin{vmatrix}
1 & 4 & 9 \\
4 & 9 & 16 \\
9 & 16 & 25 \\
\end{vmatrix}$

(d) $\begin{vmatrix}
2 & 2 & 3 \\
7 & 11 & 13 \\
17 & 19 & 23 \\
\end{vmatrix}$

3. Compute $\mathbf{a} \times \mathbf{b}$, where $\mathbf{a} = i - 2j + k$, $\mathbf{b} = 2i + j + k$.

4. Compute $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$, where $\mathbf{a}$ and $\mathbf{b}$ are as in Exercise 3 and $\mathbf{c} = 3i - j + 2k$.

5. Find the area of the parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$ given in Exercise 3.

6. A triangle has vertices $(0, 0, 0)$, $(1, 1, 1)$, and $(0, -2, 3)$. Find its area.

7. What is the volume of the parallelepiped with sides $2i + j - k$, $5i - 3k$, and $i - 2j + k$?

8. What is the volume of the parallelepiped with sides $i$, $3j - k$, and $4i + 2j - k$?

In Exercises 9 to 12, describe all unit vectors orthogonal to both of the given vectors.

9. $i$, $j$

10. $-5i + 9j - 4k$, $7i + 8j + 9k$

11. $-5i + 9j - 4k$, $7i + 8j + 9k$

12. $2i - 4j + 3k$, $-4i + 8j - 6k$

13. Compute $\mathbf{u} + \mathbf{v}$, $\mathbf{u} \cdot \mathbf{v}$, $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, and $\mathbf{u} \times \mathbf{v}$, where $\mathbf{u} = 1 - 2j + k$, $\mathbf{v} = 2i - j + 2k$.

14. Repeat Exercise 13 for $\mathbf{u} = 3i + j - k$, $\mathbf{v} = -6i - 2j - 2k$. 
15. Find an equation for the plane that
(a) is perpendicular to \( \mathbf{v} = (1, 1, 1) \) and passes through 
(1, 0, 0).
(b) is perpendicular to \( \mathbf{v} = (1, 2, 3) \) and passes through 
(1, 1, 1).
(c) is perpendicular to the line 
\( \mathbf{l}(t) = (5, 0, 2) + t(3, -1, 1) \) and passes through 
(5, -1, 0).
(d) is perpendicular to the line 
\( \mathbf{l}(t) = (-1, -2, 3) + t(0, 7, 1) \) and passes through 
(2, 4, -1).

16. Find an equation for the plane that passes through
(a) \((0, 0, 0), (2, 0, -1), \) and \((0, 4, -3)\).
(b) \((1, 2, 0), (0, 1, -2), \) and \((4, 0, 1)\).
(c) \((2, -1, 3), (0, 0, 5), \) and \((5, 7, -1)\).

17. Show that the points \((0, -2, -1), (1, 4, 0), (2, 10, 1)\) do not determine a unique plane.

18. Let \( P \) be the plane defined by the equation 
\( x + y + z = 1 \). Which of the following points are contained in \( P \)?
(a) \((0, 0, 0)\)
(b) \((1, 1, -1)\)
(c) \((-3, 8, -4)\)
(d) \((1, 2, -3)\)

19. (a) Show that two parallel planes are either identical or
they never intersect.
(b) How do two nonparallel planes intersect?

20. Find the intersection of the planes \( x + 2y + z = 0 \) and
\( x - 3y - z = 0 \).

21. Find the intersection of the planes \( x + (y - 1) + z = 0 \) and
\(-x + (y + 1) - z = 0 \).

22. Find the intersection of the two planes with equations
\( 3(x - 1) + 2y + (z + 1) = 0 \) and
\( (x - 1) + 4y - (z + 1) = 0 \).

23. (a) Prove the two triple-vector-product identities
\[ (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a} \]
and
\[ \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \]

(b) Prove \( \mathbf{u} \times \mathbf{v} \times \mathbf{w} = \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \) if and only if
\( \mathbf{u} \times \mathbf{w} \times \mathbf{v} = \mathbf{0} \)
(c) Also prove that
\( \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w}) \times \mathbf{u} + (\mathbf{w} \times \mathbf{u}) \times \mathbf{v} = \mathbf{0} \)
(called the Jacobi identity).

24. (a) Prove, without recourse to geometry, that
\[ \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = -\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v}) \]
(b) Use part (a) and Exercise 23(a) to prove that
\[ (\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u} \times \mathbf{v}') = (\mathbf{u} \cdot \mathbf{u}')(\mathbf{v} \cdot \mathbf{v}') - (\mathbf{u} \cdot \mathbf{v})(\mathbf{u}' \cdot \mathbf{v}') \]
\[ = |\mathbf{u} \cdot \mathbf{u}'| |\mathbf{v} \cdot \mathbf{v}'| \]


26. What is the geometric relation between the vectors \( \mathbf{v} \) and
\( \mathbf{w} \) if \( \|\mathbf{v} \times \mathbf{w}\| = \frac{1}{2}\|\mathbf{v}\|\|\mathbf{w}\| \)?

27. Let \( \mathbf{v} = (1, 1, 0) \) and \( \mathbf{w} = (0, 2, -1) \). Use the algebraic rules and multiplication table on page 37 to compute
\( \mathbf{v} \times \mathbf{w} \) without using determinants.

28. Find an equation for the plane that passes through the
point \((2, -1, 3)\) and is perpendicular to the line
\( \mathbf{v} = (1, -2, 2) + t(3, -2, 4) \).

29. Find an equation for the plane that passes through the
point \((1, 2, -3)\) and is perpendicular to the line
\( \mathbf{v} = (0, -2, 1) + t(1, -2, 3) \).

30. Find the equation of the line that passes through the
point \((1, -2, -3)\) and is perpendicular to the plane
\( 3x - y - 2z + 4 = 0 \).

31. Find an equation for the plane containing the two
(parallel) lines
\[ \mathbf{v}_1 = (0, 1, -2) + r(2, 3, -1) \]
and
\[ \mathbf{v}_2 = (2, -1, 0) + r(2, 3, -1). \]

32. Find a parametrization for the line perpendicular to
\((2, -1, 1)\), parallel to the plane \( 2x + y - 4z = 1 \), and
passing through the point \((1, 0, -3)\).

33. Find an equation for the plane containing the point
\((1, 0, 1)\) and the line \( \mathbf{l}(t) = (1, 2, -1) + t(1, 0, 5) \).
34. Find the distance from the point $(2, 1, -1)$ to the plane 
\[ x - 2y + 2z + 5 = 0. \]

35. Find an equation for the plane that contains the line 
\[ \mathbf{v} = (-1, 1, 2) + t(3, 2, 4) \] 
and is perpendicular to the plane 
\[ 2x + y - 3z + 4 = 0. \]

36. Find an equation for the plane that passes through 
the points $(3, 2, -1)$ and $(1, -1, 2)$ and that is parallel to the line 
\[ \mathbf{v} = (1, -1, 0) + r(3, 2, -2). \]

37. Find the moment of the force vector 
\[ \mathbf{F} = \mathbf{a} \times \mathbf{b} \] 
about the origin if the line of action is 
\[ \mathbf{a} \] 
and the direction is perpendicular to the plane of 
\[ \mathbf{a} \] 
and 
\[ \mathbf{b} \] 
respectively. (See Figure 1.3.11.)

38. Given vectors \( \mathbf{a} \) and \( \mathbf{b} \), do the equations 
\[ \mathbf{x} \times \mathbf{a} = \mathbf{b} \] 
and 
\[ \mathbf{x} \cdot \mathbf{a} = ||\mathbf{a}|| \] 
determine a unique vector \( \mathbf{x} \)? Argue both 
geometrically and analytically.

39. Determine the distance from the plane 
\[ 12x + 13y + 5z + 2 = 0 \] 
and the point $(1, 1, -5)$.

40. Find the distance from the point $(6, 1, 0)$ to the plane 
through the origin that is parallel to 
\[ \mathbf{i} - 2\mathbf{j} + \mathbf{k} \] 
and perpendicular to the plane.

41. (a) In mechanics, the moment \( \mathbf{M} \) of a force \( \mathbf{F} \) about a point \( O \) is defined to be the magnitude of \( \mathbf{F} \) times the perpendicular distance \( d \) from \( O \) to the line of action of \( \mathbf{F} \). The vector moment \( \mathbf{M} \) is the vector of magnitude \( M \) whose direction is perpendicular to the plane of \( \mathbf{O} \) and \( \mathbf{F} \), determined by the right-hand rule. Show that \( \mathbf{M} = \mathbf{R} \times \mathbf{F} \), where \( \mathbf{R} \) is any vector from \( O \) to the line of action of \( \mathbf{F} \). (See Figure 1.3.10.)

(b) Find the moment of the force vector 
\[ \mathbf{F} = \mathbf{i} - \mathbf{j} + 2\mathbf{k} \] 
newtons about the origin if the line of action is 
\[ x = 1 + t, \quad y = 1 - t, \quad z = 2t. \]

42. Show that the plane that passes through the three points 
\[ A = (a_1, a_2, a_3), \quad B = (b_1, b_2, b_3), \quad C = (c_1, c_2, c_3) \] 
consists of the points \( P = (x, y, z) \) given by

\[
\begin{vmatrix}
  a_1 - x & a_2 - y & a_3 - z \\
  b_1 - x & b_2 - y & b_3 - z \\
  c_1 - x & c_2 - y & c_3 - z \\
\end{vmatrix} = 0.
\]

(Hint: Write the determinant as a triple product.)

43. Find the distance to the line of action of 
\[ \mathbf{R} \times \mathbf{F} \] 
and the plane of \( \mathbf{O} \) and \( \mathbf{F} \). Show that 
\[ \mathbf{n}_1(\mathbf{N} \times \mathbf{a}) = \mathbf{n}_2(\mathbf{N} \times \mathbf{b}) \] 
by using Snell’s law, 
\[ \sin \theta_1 / \sin \theta_2 = n_2 / n_1, \] 
where \( \theta_1 \) and \( \theta_2 \) are the angles of incidence and refraction, respectively. (See Figure 1.3.11.)

44. Justify the steps in the following computation:

\[
\begin{vmatrix}
  1 & 2 & 3 \\
  4 & 5 & 6 \\
  7 & 8 & 10 \\
\end{vmatrix} = \begin{vmatrix}
  1 & 2 & 3 \\
  0 & -3 & -6 \\
  0 & -6 & -11 \\
\end{vmatrix} = 33 - 36 = -3.
\]

45. Show that adding a multiple of the first row of a matrix to the second row leaves the determinant unchanged; that is,

\[
\begin{vmatrix}
  a_1 + \lambda a_1 & b_1 + \lambda b_1 & c_1 + \lambda c_1 \\
  a_2 + \lambda a_2 & b_2 + \lambda b_2 & c_2 + \lambda c_2 \\
  a_3 + \lambda a_3 & b_3 + \lambda b_3 & c_3 + \lambda c_3 \\
\end{vmatrix} = \begin{vmatrix}
  a_1 & b_1 & c_1 \\
  a_2 & b_2 & c_2 \\
  a_3 & b_3 & c_3 \\
\end{vmatrix}.
\]

[In fact, adding a multiple of any row (column) of a matrix to another row (column) leaves the determinant unchanged.]

46. Suppose \( \mathbf{v}, \mathbf{w} \in \mathbb{R}^3 \) are orthogonal unit vectors. Let 
\( \mathbf{u} = \mathbf{v} \times \mathbf{w} \) Show that 
\( \mathbf{w} = \mathbf{u} \times \mathbf{v} \) and 
\( \mathbf{v} = \mathbf{w} \times \mathbf{u} \).